LOWER BOUNDS ON INTERSECTION FAMILIES FOR CERTAIN GRAPHS

PAUL HAMRICK AND GARY HU

Abstract. A family of graphs $\mathcal{F}$ is H-intersecting if the edge intersection of any two graphs in $\mathcal{F}$ contains a copy of a fixed graph H. A fundamental problem is to determine the maximum size of such a family. The trivial lower bound of $2^{\binom{n}{2}-e(H)}$ is known to be not sharp for some graphs, such as the P_4 graph, as shown by Christofides. This paper presents two main contributions. First, we introduce a general construction for H-intersecting families based on decompositions of complete multipartite graphs, giving new lower bounds for $H = K_{s_1,\ldots,s_{k-1},t}$. We compare this construction to a result by Balogh and Linz, showing that our bound is valid for a substantially wider range of parameters (beginning at $t \geq 2^{\sum_i s_i}$) and provides a stronger numerical bound for a large interval where both constructions are applicable. Second, we conjecture the $\frac{17}{128}$ Christofides bound for P_4 is optimal, which would resolve the Alon-Spencer conjecture. We computationally verify this density is optimal for families generated by connected 6-vertex host graphs with 7 or 8 edges.

1. Introduction

Definition 1.1. A family of graphs $\mathcal{F}$, whose members are subgraphs of the complete graph K_n, is said to be ***H*-intersecting** if for any two graphs $F, F' \in \mathcal{F}$, their edge intersection $F \cap F'$ contains a subgraph isomorphic to H.

One generalization of a classic result in extremal combinatorics, the Erdős-Ko-Rado Theorem [6], is to determine the maximum possible size $\max |\mathcal{F}|$ of an H-intersecting family, for different choices of H. A trivial construction for an H-intersecting family involves fixing a specific copy of H within K_n and taking all $2^{\binom{n}{2}-e(H)}$ supergraphs. This establishes a baseline lower bound of $2^{\binom{n}{2}-e(H)}$.

For some graphs, this bound is sharp. A notable example is $H = K_3$, for which a conjecture by Simonovits and Sós stating this optimality was proven by Ellis, Filmus, and Friedgut [5]. However, this trivial bound is not always optimal. For the path on four vertices, P_4, the trivial bound is $2^{\binom{n}{2}-3}$. This was also conjectured to be optimal by Simonovits and Sós [4], but disproved by Christofides in 2012 [3] with the graph shown in Figure 1.

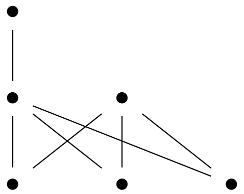

Figure 1. The 7-edge host graph on 6 vertices used in the Christofides construction [3].

Christofides identified a collection $\mathcal{G}$ of 17 subgraphs of this host graph that were pairwise P_4-intersecting. By taking all supergraphs in K_n ($n \geq 6$) for each graph in $\mathcal{G}$, he constructed a family of size $17 \cdot 2^{\binom{n}{2}-7} = \frac{17}{128} \cdot 2^{\binom{n}{2}}$. Since $\frac{17}{128} > \frac{1}{8} = 2^{-3}$, this disproved the conjecture.

Theorem 1.1 (Christofides)**.** *For $n \geq 6$, there exists a P_4-intersecting family $\mathcal{F}$ with $|\mathcal{F}| \geq \frac{17}{128} \cdot 2^{\binom{n}{2}}$.*

This method of generating a large intersecting family on K_n from a dense intersecting subfamily of a small host graph is a useful technique. If one can find a graph G with m edges and an H-intersecting family $\mathcal{G}$ of its subgraphs with density $c = |\mathcal{G}|/2^m > 2^{-e(H)}$, then for sufficiently large n, there is an H-intersecting family on K_n with size at least $c \cdot 2^{\binom{n}{2}}$.

We present a construction that improves upon the trivial bound for intersecting families where H is a complete multipartite graph. Our main result is as follows:

Theorem 1.2. *Let $k \geq 2$ be an integer, and let $s_1, \ldots, s_{k-1}$ be positive integers. Consider an integer $t \geq 2^{s_1+\cdots+s_{k-1}}$, and let $H = K_{s_1,\ldots,s_{k-1},t}$ be a complete k-partite graph with parts of sizes $s_1, \ldots, s_{k-1}$, and t. Denote by $N = |E(H)|$ the number of edges in H. Then, for any integer $n \geq s_1 + \cdots + s_{k-1} + t + 2$, there exists an H-intersecting family of subgraphs with size at least*

$$\frac{(t+2)(2^{s_1+\cdots+s_{k-1}} - 1) + 1}{2^{|E(K_{s_1,\ldots,s_{k-1},t+2})|}} 2^{\binom{n}{2}} > \frac{1}{2^N} 2^{\binom{n}{2}}.$$

An alternative construction for non-trivial H-intersecting families, where $H = K_{s_1,\ldots,s_{k-1},t}$, was recently presented in [2] by Balogh and Linz. We do a comparison between the two results in Section 4.

2. Construction for Theorem 1.2

Let the host graph be $G = K_{s_1,\ldots,s_{k-1},t+2}$. The vertex set of G is partitioned into sets $V_1, \ldots, V_{k-1}, W$ with $|V_i| = s_i$ and $|W| = t+2$. The graph H is an induced subgraph of G. Consider the $t+2$ subgraphs of G of the form $G_w = G[V(G) \setminus \{w\}]$ for each $w \in W$. Each G_w is isomorphic to $K_{s_1,\ldots,s_{k-1},t+1}$. The intersection of any two distinct subgraphs, $G_w \cap G_{w'}$, is isomorphic to $K_{s_1,\ldots,s_{k-1},t}$, which is H.

We construct an H-intersecting family $\mathcal{G}$ of subgraphs of G as follows. Let $m = s_1 + \cdots + s_{k-1}$. For each of the $t+2$ subgraphs G_w, we take all supergraphs of G_w within G except for G itself. This gives $2^{e(G)-e(G_w)} - 1 = 2^m - 1$ distinct subgraphs for each w. Adding the complete graph G to this collection gives a family of size $|\mathcal{G}| = (t+2)(2^m - 1) + 1$. Since the intersection of any two chosen subgraphs must contain some $G_w \cap G_{w'}$ or G_w itself, this family $\mathcal{G}$ is H-intersecting.

To obtain the family $\mathcal{F}$ on K_n, we take all supergraphs of the members of $\mathcal{G}$. The size of this family is $|\mathcal{G}| \cdot 2^{-e(G)}$. We must show this improves the trivial bound:

$$|\mathcal{G}| \cdot 2^{-e(G)} > 2^{-e(H)} \iff |\mathcal{G}| > 2^{e(G)-e(H)}$$

The difference in edges is $e(G) - e(H) = m(t+2) - mt = 2m$. It remains to check $(t+2)(2^m - 1) + 1 > 2^{2m}$. Given the condition $t \geq 2^m$, we have:

$$(t+2)(2^m - 1) + 1 \geq (2^m + 2)(2^m - 1) + 1 = (2^{2m} + 2^m - 2) + 1 = 2^{2m} + 2^m - 1$$

Since $m \geq 1$, this is strictly greater than 2^{2m}, which completes the proof.

3. A Corollary and a Generalization

In particular, if $k = 2$, this gives an improvement on the trivial bound for certain bipartite graphs.

Corollary 3.1. *If $s \in \mathbb{N}_{>0}$, $t \geq 2^s$, and $n \geq s+t+2$, there exists a $K_{s,t}$-intersecting family of size at least*

$$\frac{(t+2)(2^s-1)+1}{2^{s(t+2)}} 2^{\binom{n}{2}} > \frac{1}{2^{st}} 2^{\binom{n}{2}}.$$

The proof technique can be generalized as follows.

Proposition 3.2. *Let H be a graph. Suppose there exists a host graph G and a set of N distinct subgraphs $\{H_1, \dots, H_N\}$ of G that satisfy the following conditions:*

(1) *Intersection Property: For any $i, j \in \{1, \dots, N\}$, the intersection $H_i \cap H_j$ contains a subgraph isomorphic to H.*
(2) *Disjoint Complement Property: For any $i \neq j$, the union $E(H_i) \cup E(H_j) = E(G)$.*

Then for any integer $n \geq |V(G)|$, there exists an H-intersecting family $\mathcal{F}$ of subgraphs of K_n of size

$$\left(1 + \sum_{i=1}^{N} \left(2^{|E(G)|-|E(H_i)|} - 1\right)\right) \cdot 2^{\binom{n}{2} - |E(G)|}.$$

Proof. Let $\mathcal{G}$ be a family of subgraphs of G defined as follows:

$$\mathcal{G} = \{G\} \cup \bigcup_{i=1}^{N} \{F \mid E(H_i) \subseteq E(F) \text{ and } F \neq G\}.$$

This family consists of the graph G itself, along with every proper supergraph of each H_i.

Now, we show that the family $\mathcal{G}$ is H-intersecting. Let F_1, F_2 be two distinct members of $\mathcal{G}$. We must show that $F_1 \cap F_2$ contains a subgraph isomorphic to H. Any element of $\mathcal{G}$ contains some H_i as a subgraph, so let F_1 have H_{i_1} as a subgraph and F_2 have H_{i_2} as a subgraph. Then $F_1 \cap F_2$ contains $H_{i_1} \cap H_{i_2}$, which we know contains a subgraph isomorphic to H. Thus, $\mathcal{G}$ is an H-intersecting family.

Next, we show that the size of $\mathcal{G}$ is $1 + \sum_{i=1}^{N} \left(2^{|E(G)|-|E(H_i)|} - 1\right)$. Let $\mathcal{S}_i = \{F \mid E(H_i) \subseteq E(F) \text{ and } F \neq G\}$. Then $\mathcal{G} = \{G\} \cup \bigcup_{i=1}^{N} \mathcal{S}_i$. To calculate $|\mathcal{G}|$, we show that the sets $\mathcal{S}_i$ are pairwise disjoint for $i \neq j$. Assume for contradiction that there exists a graph $F \in \mathcal{S}_i \cap \mathcal{S}_j$ for $i \neq j$. By definition of $\mathcal{S}_i$ and $\mathcal{S}_j$, it must be that $E(H_i) \subseteq E(F)$ and $E(H_j) \subseteq E(F)$. This implies $E(H_i) \cup E(H_j) \subseteq E(F)$. From Property 2, we know $E(H_i) \cup E(H_j) = E(G)$. Therefore, $E(G) \subseteq E(F)$. Since F is a subgraph of G, we must have $E(F) = E(G)$, which implies $F = G$. This contradicts the fact that the sets $\mathcal{S}_i$ and $\mathcal{S}_j$ exclude G. The sets are thus pairwise disjoint. The size of $\mathcal{G}$ is

$$|\mathcal{G}| = |\{G\}| + \sum_{i=1}^{N} |\mathcal{S}_i| = 1 + \sum_{i=1}^{N} \left(2^{|E(G)|-|E(H_i)|} - 1\right).$$

Finally, we construct the family $\mathcal{F}$ on the n vertices of K_n. We embed G in K_n. The family $\mathcal{F}$ is the set of all supergraphs in K_n of the members of $\mathcal{G}$. Each graph $F_G \in \mathcal{G}$ can be extended to a supergraph in K_n by adding any subset of the $\binom{n}{2} - |E(G)|$ edges that exist in K_n but not in G. Since all members of $\mathcal{G}$ are distinct, their corresponding sets of supergraphs in K_n are also disjoint.

The total size of $\mathcal{F}$ is therefore $|\mathcal{G}| \cdot 2^{\binom{n}{2} - |E(G)|}$. Substituting the expression for $|\mathcal{G}|$ gives the stated result.

□

4. COMPARISON TO THE BALOGH-LINZ BOUND

Here, we compare the lower bound in 1.2 with the one by Balogh-Linz mentioned in the introduction [2]. As before, let $H = K_{s_1,\ldots,s_r,t}$ be the target graph and let $S = \sum_{i=1}^{r} s_i$. The trivial bound for an H-intersecting family is $2^{\binom{n}{2}-|E(H)|}$. The Balogh-Linz result is as follows.

Theorem 4.1 (Balogh-Linz Bound, [2])**.** *Let* $s_1, s_2, \ldots, s_r, t$ *be integers with* $s_i \geq 1$ *for* $1 \leq i \leq r$ *and* $t > 2^{2\sum_i s_i} - 2\sum_i s_i - 1$. *Then there exists a* $K_{s_1,\ldots,s_r,t}$*-intersecting family of graphs* $\mathcal{H}$ *with*

$$|\mathcal{H}| > 2^{\binom{n}{2}-\sum_{1\leq i<j\leq r} s_i s_j - \sum_{i=1}^{r} s_i t}.$$

The construction in [2] establishes a family whose size is $(t + 2S + 1)^S \cdot 2^{\binom{n}{2}-|E(H)|-2S^2}$. The improvement factor over the trivial bound, which we denote $f_{BL}(t)$, is therefore:

$$f_{BL}(t) = \frac{(t + 2S + 1)^S}{2^{2S^2}}.$$

Note that $f_{BL}(t)$ is a polynomial in t of degree S.

In contrast, our construction from Theorem 1.2 provides a family with a size, linear in t, of $((t + 2)(2^S - 1) + 1) \cdot 2^{\binom{n}{2}-|E(H)|-2S}$. This gives a linear improvement factor, which we denote $f_{\text{new}}(t)$:

$$f_{\text{new}}(t) = \frac{(t + 2)(2^S - 1) + 1}{2^{2S}}.$$

This distinction in the degree of t implies that the Balogh-Linz construction is asymptotically stronger for sufficiently large t. Simultaneously, an analysis of the respective conditions on t reveals that our construction is valid across a wider range of parameters.

For example, the Balogh-Linz construction requires $t > 2^{2S} - 2S - 1$. Our construction requires only $t \geq 2^S$. This creates a substantial interval $t \in [2^S, 2^{2S} - 2S - 1]$ where our construction is the only one applicable.

For $t \geq 2^{2S} - 2S$, both constructions are valid. We can compute the crossover point, t_{cross}, where $f_{BL}(t)$ becomes larger than $f_{\text{new}}(t)$.

Table 1 provides a comprehensive numerical analysis for small values of S. All crossover points are computed by solving $f_{BL}(t) > f_{\text{new}}(t)$ for the smallest integer t.

TABLE 1. Comparison of applicability and performance of $f_{\text{new}}(t)$ and $f_{BL}(t)$.

S	$f_{\text{new}}(t)$ Validity (Our Bound)	$f_{BL}(t)$ Validity (Balogh-Linz Bound)	Validity Gap (Our Bound Only)	Crossover t_{cross} (where $f_{BL} > f_{\text{new}}$)
2	$t \geq 4$	$t \geq 12$	$[4, 11]$	$t = 41$
3	$t \geq 8$	$t \geq 58$	$[8, 57]$	$t = 160$
4	$t \geq 16$	$t \geq 248$	$[16, 247]$	$t = 621$
5	$t \geq 32$	$t \geq 1014$	$[32, 1013]$	$t = 2404$

The data in Table 1 leads to two conclusions:

- The validity gap, where our construction is the only one known to improve the trivial bound, grows doubly exponentially with S. For $S = 5$, our bound applies to a range of 982 values of t for which the Balogh-Linz construction is not valid.

- Even after the Balogh-Linz construction becomes valid, our linear bound remains numerically superior for a significant subsequent interval. For example, at the first point of Balogh-Linz validity for $S = 5$ (i.e., $t = 1014$), our improvement factor is $f_{\text{new}}(1014) \approx 30.76$, whereas the Balogh-Linz factor is $f_{BL}(1014) = (1025/1024)^5 \approx 1.005$.

We can estimate the upper point of this range by equating the two improvement factors:

$$\frac{t}{2^S} \approx f_{\text{new}}(t) \approx f_{BL}(t) \approx \frac{t^S}{2^{2S^2}}.$$

Rearranging gives $t^{S-1} \approx 2^{2S^2-S}$, or $t \approx 2^{2S+1+\frac{1}{S-1}}$. This approximation aligns closely with the actual crossover points; for $S = 5$, it gives $t \approx 2^{11+\frac{1}{4}} \approx 2435$, which is close to the actual value of 2404. Thus, our construction typically improves on the Balogh-Linz construction for t up to approximately 2^{2S+1}.

In summary, while the Balogh-Linz construction is asymptotically more powerful, Theorem 1.2 provides a bound that is effective for a wider range of parameters. It exclusively improves the trivial bound in the interval $t \in [2^S, 2^{2S} - 2S - 1]$ and remains numerically stronger for a large interval thereafter.

5. APPLICATIONS AND NEW BOUNDS

Theorem 1.2 provides a rich source of new bounds for intersecting families where the target graph is not a star-forest. We select the smallest integer t satisfying the theorem's condition, $t = 2^s$, to maximize the improvement over the trivial bound.

5.1. **Complete Bipartite Graphs ($H = K_{s,t}$).** The case $k = 2$ gives bounds for complete bipartite graphs. Here $s_1 = s$, so we set $S = s$. We select the smallest integer t satisfying the theorem's condition, $t = 2^S = 2^s$.

Corollary 5.1. *The following improved lower bounds for $K_{s,2^s}$-intersecting families hold:*

1. *For $H = K_{2,4}$ ($n \geq 8$): $|\mathcal{F}| \geq \frac{19}{2^{12}} \cdot 2^{\binom{n}{2}}$ (vs. trivial $\frac{16}{2^{12}}$).*
2. *For $H = K_{3,8}$ ($n \geq 13$): $|\mathcal{F}| \geq \frac{71}{2^{30}} \cdot 2^{\binom{n}{2}}$ (vs. trivial $\frac{64}{2^{30}}$).*
3. *For $H = K_{4,16}$ ($n \geq 22$): $|\mathcal{F}| \geq \frac{271}{2^{72}} \cdot 2^{\binom{n}{2}}$ (vs. trivial $\frac{256}{2^{72}}$).*
4. *For $H = K_{5,32}$ ($n \geq 39$): $|\mathcal{F}| \geq \frac{1055}{2^{170}} \cdot 2^{\binom{n}{2}}$ (vs. trivial $\frac{1024}{2^{170}}$).*

Proof. We prove (4) as an example. Let $s = 5$, so $t = 2^5 = 32$. The target is $H = K_{5,32}$ with $|E(H)| = 160$. The host graph is $G = K_{5,34}$ with $|E(G)| = 170$. The density factor from our construction is

$$\frac{(t+2)(2^s-1)+1}{2^{|E(G)|}} = \frac{(32+2)(2^5-1)+1}{2^{170}} = \frac{1055}{2^{170}}.$$

The trivial bound gives a density of $\frac{1}{2^{160}} = \frac{1024}{2^{170}}$. Since $1055 > 1024$, our bound is better. The other proofs are analogous. ☐

5.2. **General Multipartite Graphs.** The construction is equally effective for multipartite graphs with more than two parts.

Corollary 5.2. *The following improved lower bounds for multipartite-intersecting families hold:*

1. *For $H = K_{1,1,4}$ ($n \geq 8$): $|\mathcal{F}| \geq \frac{19}{2^{13}} \cdot 2^{\binom{n}{2}}$ (vs. trivial $\frac{16}{2^{13}}$).*
2. *For $H = K_{1,2,8}$ ($n \geq 13$): $|\mathcal{F}| \geq \frac{71}{2^{32}} \cdot 2^{\binom{n}{2}}$ (vs. trivial $\frac{64}{2^{32}}$).*

(3) *For* $H = K_{2,2,16}$ *(*$n \geq 22$*):* $|\mathcal{F}| \geq \frac{271}{2^{76}} \cdot 2^{\binom{n}{2}}$ *(vs. trivial* $\frac{256}{2^{76}}$*).*

(4) *For* $H = K_{1,1,1,8}$ *(*$n \geq 13$*):* $|\mathcal{F}| \geq \frac{71}{2^{33}} \cdot 2^{\binom{n}{2}}$ *(vs. trivial* $\frac{64}{2^{33}}$*).*

Proof. We prove (3) as an example. Let $k = 3$ with $s_1 = 2, s_2 = 2$. Then $S = 4$ and we set $t = 2^4 = 16$. The target is $H = K_{2,2,16}$ with $|E(H)| = 2 \cdot 2 + 2 \cdot 16 + 2 \cdot 16 = 68$. The host is $G = K_{2,2,18}$ with $|E(G)| = 2 \cdot 2 + 2 \cdot 18 + 2 \cdot 18 = 76$. The new density is

$$\frac{(t+2)(2^S - 1) + 1}{2^{|E(G)|}} = \frac{(16+2)(2^4 - 1) + 1}{2^{76}} = \frac{18 \cdot 15 + 1}{2^{76}} = \frac{271}{2^{76}}.$$

The trivial bound gives a density of $\frac{1}{2^{68}} = \frac{256}{2^{76}}$. ☐

6. On the Optimality of the P_4 Bound

While the Christofides construction improved the known bound for P_4-intersecting families from the trivial $\frac{1}{8}$ to $\frac{17}{128}$, its optimality remains an open question. It is believed among some experts in the field that this bound is indeed sharp, though this conjecture does not appear to be formally stated in the literature. We state it here.

Conjecture 6.1. *For* $n \geq 6$*, the maximum size of a* P_4*-intersecting family is* $\frac{17}{128} \cdot 2^{\binom{n}{2}}$*. Furthermore, any extremal family is isomorphic to one generated by the Christofides construction.*

This conjecture is particularly significant due to its connection to a more general conjecture proposed by Alon and Spencer [1].

Conjecture 6.2 (Alon-Spencer [1])**.** *There exists some constant* $\epsilon > 0$ *such that if* H *is a fixed graph that is not a star-forest and* $\mathcal{F}$ *is an* H*-intersecting family on* K_t*, then*

$$|\mathcal{F}| < \left(\frac{1}{2} - \epsilon\right) 2^{\binom{t}{2}}.$$

In fact, the conjecture of Alon and Spencer would follow from 6.1. The reasoning is that any graph H that is not a star-forest must contain K_3 or P_4 as a subgraph, and either condition would imply a bound of at most $\frac{17}{128} \cdot 2^{\binom{n}{2}}$.

- If H contains K_3, any H-intersecting family $\mathcal{F}$ is also a K_3-intersecting family. Thus, $|\mathcal{F}|$ is bounded by the known maximum size for K_3, which is $\frac{1}{8} \cdot 2^{\binom{n}{2}}$, as proved by Ellis, Filmus, and Friedgut [5].
- If H is triangle-free but contains P_4, any H-intersecting family $\mathcal{F}$ is also a P_4-intersecting family. If Conjecture 6.1 holds, $|\mathcal{F}|$ would be bounded by $\frac{17}{128} \cdot 2^{\binom{n}{2}}$.

As partial evidence supporting Conjecture 6.1, we performed a computational search for denser constructions on small host graphs. This search yielded no constructions outperforming the Christofides bound. We summarize our findings as follows.

Theorem 6.3. *The Christofides density of* $\frac{17}{128}$ *is optimal for all* P_4*-intersecting families generated from a connected host graph* G *on 6 vertices with 7 or 8 edges.*

Proof. Our method involved an exhaustive search using Python (our implementation is available in Appendix A). For a given host graph G, we generated its subgraphs with at least 3 edges (the number of edges in P_4) and constructed an auxiliary graph whose vertices represented these subgraphs. An edge was placed between two vertices in the auxiliary graph if their corresponding subgraphs had a P_4 in their intersection. The size of the largest P_4-intersecting family supported by G is then the size of the maximum clique in this auxiliary graph, a problem known to be NP-complete [7]. $\square$

While our search was limited to small graphs, its results support the conjecture that the Christofides density is optimal.

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APPENDIX A. CODE 1

```
import networkx as nx
import itertools
import matplotlib.pyplot as plt

# Step 1: Create a function that checks if two graphs share a P3

def is_edge(i, j, input) -> bool:
    G1 = input[i]
    G2 = input[j]
    R = nx.intersection(G1, G2)
    my_nodes = R.nodes()
    perms = itertools.permutations(my_nodes, 6)
    for perm in perms:
        if R.has_edge(perm[0], perm[1]):
            if R.has_edge(perm[1], perm[2]):
                if R.has_edge(perm[2], perm[3]):
                    return True
    return False
```

```
# Main function: finds the biggest collection of subgraphs in which any two share a P3
def compute(K, M, almost) -> int:
    edges = list(K.edges())
    n = len(edges)
    # Enumerate all 2^n subgraphs
    subgraphs = []
    for bits in range(2**n):
        bits_str = format(bits, f'0{n}b')
        G = nx.Graph()
        G.add_nodes_from(K)
        G.add_edges_from([edges[i] for i in range(n) if bits_str[i] == '1'])
        if (len(list(G.edges())) >= M):
            subgraphs.append(G)

    # Make a graph connecting subgraphs if their intersection includes a copy of P3
    Supergraph = nx.Graph()
    size = len(subgraphs)
    Supergraph.add_nodes_from(range(size))

    for i in range(size):
        j = i + 1
        while (j < size):
            if is_edge(i, j, subgraphs):
                Supergraph.add_edge(i, j)
            j = j + 1
    top = max(len(c) for c in nx.find_cliques(Supergraph))
    if almost: # Adding an option to draw all the subgraphs in the maximum clique
        for c in nx.find_cliques(Supergraph):
            t = len(c)
            if (t == top):
                for i in c:
                   nx.draw(subgraphs[i], with_labels=True)
                   plt.show()
    return top

# Start with a complete graph on 6 vertices
J = nx.complete_graph(6)
e = 7 # number of edges to remove
edges = list(J.edges())
edge_lists = itertools.combinations(edges, e)
i = 0
# Iterate over graphs with the right number of edges
for edge_list in edge_lists:
    K = nx.complete_graph(6)
    K.remove_edges_from(edge_list)
    if nx.is_connected(K):
        big = compute(K, 4, False)
        print(big)
        if (big > 34): # If the max clique > Christofides graph
            nx.draw(K)
            plt.show()
```

Department of Mathematics, University of North Carolina at Chapel Hill, Chapel Hill, NC 27599
Email address: `hamri@unc.edu`

Department of Mathematics, Williams College, Williamstown, MA 01267
Email address: `gh7@williams.edu`