# Projective Toric Surfaces via Convex Lattice Polygons

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## 1 Introduction

Throughout this course, we have developed an algebra–geometry dictionary that connects systems of polynomial equations with affine varieties. In Chapter 8, we extended this perspective by introducing projective varieties, obtained by adding points at infinity. This allowed us to study varieties in projective space $\mathbb{P}^n$, where they are defined by homogeneous ideals.

Everyone has their own favorite and least favorite parts of the course. Personally, I found parametrizations the most challenging, since there were so many variables in play at once. I often wished for a simpler, less messy approach. As it turns out, toric varieties provide exactly that! One of the reasons toric varieties are useful is that they contain dense open subsets that can be parameterized in a very natural way using monomials.

In this writeup, we focus on projective toric surfaces, which are 2-dimensional toric varieties. One important consequence of this monomial parameterization is that projective toric surfaces can be constructed directly from convex lattice polygons whose vertices have integer coordinates. This gives a convenient combinatorial framework to work with and leads to many explicit examples that are well suited for concrete computations, both by hand and with computer algebra systems.

The main goal of this writeup is to build on what we have learned this semester to guide the reader from the basic construction of projective toric surfaces to understand how the homogeneous binomial equations defining these surfaces can be computed explicitly by finding the kernel of an integer matrix derived from a 2-dimensional lattice polygon. In doing so, we provide a solution to Project 15 in [1], following the proof in [2].

## 2 Preliminary Definitions

Let's briefly recall some things we learned at the end of the semester. Recall that we defined the $n$-dimensional projective space $\mathbb{P}^n$ by identifying points $x$ and $y$ in $\mathbb{C}^{n+1}\backslash\{0\}$ if there exists a nonzero scalar $\lambda \in \mathbb{C}$ such that $x = \lambda y$. And we also have an equivalence class of points denoted by homogeneous coordinates $[x_0 : x_1 : ... : x_n]$. Because the coordinates of a point in $\mathbb{P}^n$ are only determined up to a nonzero scalar multiple, arbitrary polynomials do not have well-defined zero sets in projective space. But if we restrict our attention to homogeneous polynomials (where every term has the same total degree), the zero set becomes well-defined. For a finite set of homogeneous polynomials $F_1, ..., F_k \in \mathbb{C}[x_0, ..., x_n]$, we can now define a projective variety as the set of their simultaneous zeroes:

$$V(F_1, ..., F_k) = \{[p] \in \mathbb{P}^n | F_1(p) = ... = F_k(p) = 0\}.$$

Conversely, given a projective variety $X \subset \mathbb{P}^n$, we define its ideal $I(X)$ to consist of all homogeneous polynomials $F \in \mathbb{C}[x_0, ..., x_n]$ that vanish on $X$, meaning $F(p) = 0$ for all $[p] \in X$.

When working with parameterizations, we frequently encounter sets of points in $\mathbb{P}^n$ that are not varieties themselves. To associate a variety to an arbitrary set $X \subset \mathbb{P}^n$, we defined the Zariski closure of $X$, denoted

$\overline{X}$, as the smallest variety containing $X$. As the name suggests, this is the closure of the set $X$ in the Zariski topology.

Recall that a standard monomial in the variables $x_1, ..., x_n$ is an expression of the form $x^a = x_1^{a_1} x_2^{a_2} ... x_n^{a_n}$, where the exponents $a_i$ are nonnegative integer. If we expand this definition to allow the exponents $a_i$ to be negative integers, we call the resulting expression $x^a$ a Laurent monomial. Because Laurent monomials can have negative exponents, evaluating them requires that our variables do not take the value of zero, or else we have problems with division by zero. The workaround is to let $\mathbb{C}^* = \mathbb{C}\backslash\{0\}$ denote the set of nonzero complex numbers. Then we define the 2-dimensional algebraic torus as the set $(\mathbb{C}^*)^2$. As we will see in the next section, the algebraic torus is the domain for the parameterizations that construct 2-dimensional toric surfaces.

## 3 Constructing Projective Toric Surfaces

To define a projective toric surface, we begin with a purely combinatorial object: a convex lattice polygon. Let $P \subset \mathbb{R}^2$ be a convex polygon whose vertices are elements of the integer lattice $\mathbb{Z}^2$. Because $P$ is convex, it contains its interior as well as its boundary. We are interested in the finite set of all integer lattice points contained within $P$, which we denote as $P \cap \mathbb{Z}^2 = \{m_0, m_1, ..., m_n\}$.

Using these $n+1$ lattice points, we can then define a parameterization map into $n$-dimensional projective space $\mathbb{P}^n$, and take our domain to be the 2-dimensional algebraic torus $T = (\mathbb{C}^*)^2$. Define the map $\phi_p : T \to \mathbb{P}^n$ by evaluating Laurent monomials corresponding to the lattice points:

$$\phi_P(t) = [t^{m_0} : t^{m_1} : ... : t^{m_n}].$$

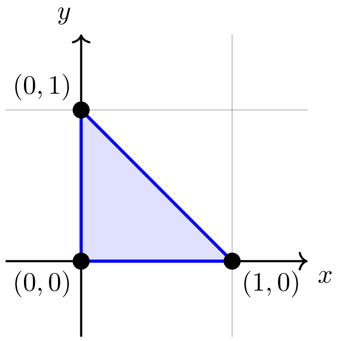
To understand this notation, if a lattice point is given by $m_i = (a, b)$, the expression $t^{m_i}$ is the monomial $t_1^a t_2^b$. Since the integer coordinates $a$ and $b$ can be negative, it is important that we evaluate this map only for nonzero complex numbers $t_1$ and $t_2$, which is why we restricted our domain to the algebraic torus. Additionally, translating the polygon so that one of its lattice points (say $m_0$) lies at the origin $(0, 0)$, we can make the first coordinate in our projective mapping $t^{0,0} = 1$, which means our image lies inside the standard affine open subset $U_0 \subset \mathbb{P}^n$.

The image of $\phi_p(t)$ gives us a parameterized surface in $\mathbb{P}^n$. This image might not be a projective variety since it might not be Zariski closed, but can just take the Zariski closure $\overline{\text{Im}\phi_P}$. This leads us to our most important definition:

**Definition 3.1.** *The **toric surface** $X_P$ associated to a 2-dimensional convex polygon $P$ is the Zariski closure of the image of $\phi_P$ in $\mathbb{P}^n$.*

Now let's look at two examples to see how this correspondence between polygons and varieties works.

**Example 3.2** (The Unit Triangle)**.** *Let $P_1$ be the right triangle with vertices $(0, 0)$, $(1, 0)$, and $(0, 1)$.*



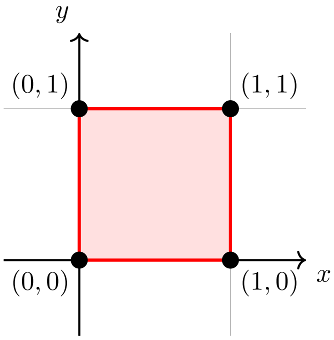
*The intersection $P_1 \cap \mathbb{Z}^2$ is these three vertices, so the corresponding parameterization map $\phi_{P_1} : T \to \mathbb{P}^2$ is given by evaluating the monomials $t_1^0 t_2^0$, $t_1^1 t_2^0$, and $t_1^0 t_2^1$:*

$$\phi_{P_1}(t) = [1 : t_1 : t_2]$$

*This map parameterizes the open subset of $\mathbb{P}^2$ where no coordinate is zero. Taking the Zariski closure adds the missing points at infinity (which geometrically correspond to the coordinate axes), so the resulting toric surface is*

$$X_{P_1} = \overline{\operatorname{Im}(\phi_{P_1})} = \mathbb{P}^2.$$

**Example 3.3** (The Unit Square)**.** *Let $P_4$ be the square with vertices $(0,0)$, $(1,0)$, $(0,1)$, and $(1,1)$.*



*Then the lattice points are the four vertices, and the map into $\mathbb{P}^3$ is*

$$\phi_{P_4}(t) = [1 : t_1 : t_2 : t_1 t_2].$$

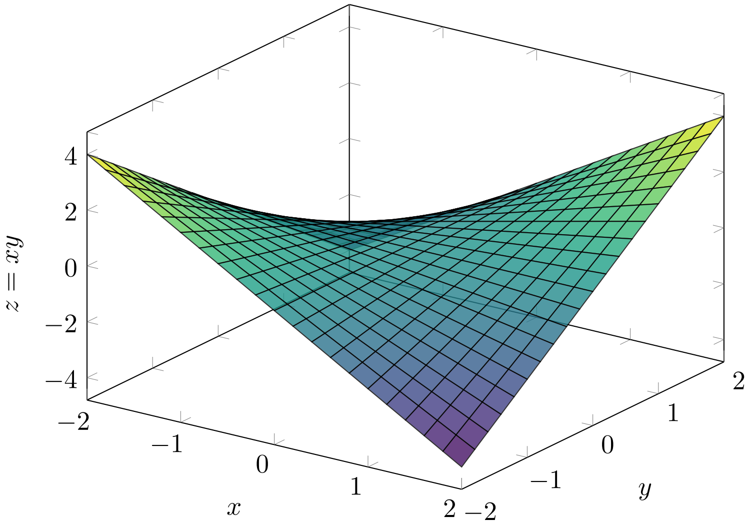
*To understand the geometry of this map, let $[x_0 : x_1 : x_2 : x_3]$ be the homogeneous coordinates of $\mathbb{P}^3$. The coordinates of our parameterized points are given by*

$$x_0 = t^0 = 1, \quad x_1 = t_1, \quad x_2 = t_2, \quad x_3 = t_1 t_2,$$

*hence*

$$x_0 x_3 - x_1 x_2 = (1)(t_1 t_2) - (t_1)(t_2) = 0.$$

*We can visualize this by looking at the real points in the affine patch where $x_0 = 1$. Here, the image of $\phi_{P_4}$ consists of points $(1, x, y, z)$ where $z = xy$. Thus, the real image is the graph of the saddle surface $f(x, y) = xy$ in $\mathbb{R}^3$, minus the coordinate axes (since $t_1, t_2 \neq 0$).*



*The toric surface $X_{P_4}$ is the Zariski closure of this map, which fills in those missing limit points to give the quadric surface defined by $x_0 x_3 - x_1 x_2 = 0$.*

So far we have a nice parameterization that gives a geometric and combinatorial way to construct these surfaces, but still this is an explicit representation. To understand $X_P$ as a variety, we need to solve the implicitization problem and find the homogeneous polynomial equations that define it. Next, we will see that the combinatorial data of the polygon gives us an ideal generated by binomials.

## 4 The Defining Ideal of an Affine Toric Variety

Now we discuss the implicitization problem, and find the homogeneous polynomial equations that define the projective toric surface $X_P$. But before we do this, it's instructive to first look at the affine case.

Let $A = [m_1 \dots m_n]$ be an $m \times n$ matrix with integer entries. This defines a parameterization map $\phi_A : (\mathbb{C}^*)^m \to \mathbb{C}^n$ given by evaluating Laurent monomials

$$\phi_A(t) = (t^{m_1}, \dots, t^{m_n})$$

where $t = (t_1, \dots, t_m)$. The affine toric variety $Y_A$ is the Zariski closure of the image of this map.

Consider a monomial $x^u = x_1^{u_1} \dots x_n^{u_n}$ where $u \in \mathbb{N}^n$. If we substitute $\phi_A(t)$ into $x^u$, we get $t^{Au}$. So if we want to find a binomial $f(x) = x^u - x^v$ that vanishes on $Y_A$, we need $f(\phi_A(t)) = t^{Au} - t^{Av}$ to be zero for all $t \in (\mathbb{C}^*)^m$. This happens if and only if $Au = Av$, which means $u - v$ is in the kernel of $A$. This motivates the definition of the toric ideal associated to $A$:

$$I_A = \langle x^u - x^v \mid u - v \in \ker A, \text{ and } u, v \in \mathbb{N}^n \rangle$$

**Theorem 4.1.** *Let $A$ be an integer matrix. Then the ideal of the affine toric variety $Y_A$ is $I(Y_A) = I_A$.*

*Proof.* We show equality $I(Y_A) = I_A$ by showing inclusion in both directions.

**Claim**: $I_A \subseteq I(Y_A)$.

We need to show that the binomial generators of $I_A$ vanish on the variety. Take any $u, v \in \mathbb{N}^n$ such that $u - v \in \ker(A)$. By definition, this means $Au = Av$. Let $f(x) = x^u - x^v$. We substitute our parameterization into $f$:

$$f(\phi_A(t)) = t^{Au} - t^{Av} = 0$$

Since the polynomial $f$ evaluates to zero for all $t$, the variety $V(f)$ contains the entire image $\mathrm{Im}(\phi_A)$. Since the Zariski closure $Y_A$ is the smallest variety containing $\mathrm{Im}(\phi_A)$, $V(f)$ must also contain $Y_A$. Therefore $f \in I(Y_A)$, which proves $I_A \subseteq I(Y_A)$.

**Claim**: $I(Y_A) \subseteq I_A$.

Let $U$ be the set of all polynomials that are in $I(Y_A)$ but are not in $I_A$. We want to show that $U$ is empty. Assume $U$ is nonempty. Fix a monomial order on $\mathbb{C}[x_1, \dots, x_n]$, for example lexicographic order, where $x^a > x^b$ if the first nonzero coordinate of $a - b$ is positive. Since the lexicographic order is a well-ordering, we can pick a polynomial $f \in U$ that has the smallest leading term among all polynomials in $U$. Let this leading term be $cx^u$. Since $f \in I(Y_A)$, it must vanish on the parameterization, so we must have $f(\phi_A(t)) = 0$. When we evaluate $f$, the leading term becomes $ct^{Au}$. For our entire polynomial to evaluate to zero, this term $ct^{Au}$ must be cancelled out by the evaluation of other terms in $f$, so $f$ must contain at least one other term, say $c'x^v$, such that its evaluation produces a scalar multiple of $t^{Av}$ where $Au = Av$.

Because $Au = Av$, the difference $u - v$ is in the kernel of $A$, so the binomial $x^u - x^v$ is an element of $I_A$. We can then use this binomial to reduce $f$ by defining a new polynomial:

$$g(x) = f(x) - c(x^u - x^v)$$

By subtracting $cx^u$, we have cancelled out the original leading term of $f$. Since $x^v$ was already a monomial appearing in $f$, subtracting it does not introduce any monomials that are lexicographically larger than $x^u$. Therefore the leading term of $g(x)$ is smaller than the leading term of $f(x)$.

Now is $g$ in the set $U$?

- If $g \in U$, we have found a polynomial in $U$ with a smaller leading term than $f$, contradicting our choice of $f$ as the minimal element.
- If $g \notin U$, it means $g \in I_A$. But notice $f(x) = g(x) + c(x^u - x^v)$. Since $g \in I_A$ and $(x^u - x^v) \in I_A$, their sum $f(x)$ must also be in $I_A$. Therefore $f \notin U$, which contradicts the fact that we chose $f$ from $U$ to begin with.

Since both cases lead to a contradiction, the set $U$ must be empty. Therefore every polynomial in $I(Y_A)$ is also in $I_A$, so $I(Y_A) \subseteq I_A$. This finishes the proof of the affine case. $\square$

## 5 The Defining Ideal of a Projective Toric Variety

Now we return to the projective toric surface $X_P$ constructed from a 2-dimensional lattice polygon $P$. For a binomial $f(x) = x^u - x^v$ to have a well-defined zero set in projective space $\mathbb{P}^n$, it needs to be homogeneous. This requires the sum of the coordinates of $u$ to equal the sum of the coordinates of $v$, which is equivalent to saying that the vector $u - v$ is orthogonal to the vector $(1, 1, \ldots, 1)$.

To ensure homogeneity, we adjoin a row of ones to our matrix. If $P \cap \mathbb{Z}^2 = \{m_0, \ldots, m_n\}$, we define $A_P$ to be the matrix whose columns are $a_i = (1, m_i)^T$. By forcing $u - v \in \ker(A_P)$, the top row of ones makes $u - v$ orthogonal to $(1, 1, \ldots, 1)$, so $I_{A_P}$ is generated by homogeneous binomials.

To see how the kernel computation gives the defining equation, consider the matrix for the unit square $P_4$ from Example 3.3:

$$A_{P_4} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

We look for vectors in the kernel, meaning $A_{P_4}(u - v) = 0$ for $u, v \in \mathbb{N}^4$. The vector $(1, -1, -1, 1)^T$ is in the kernel, which we can write as $u - v$ with $u = (1, 0, 0, 1)^T$ and $v = (0, 1, 1, 0)^T$. This corresponds to the binomial generator:

$$x^u - x^v = x_0 x_3 - x_1 x_2.$$

**Theorem 5.1.** *Let $A_P$ be the homogenized integer matrix derived from a lattice polygon $P$. Then the homogeneous ideal of the projective toric variety $X_P$ is $I(X_P) = I_{A_P}$.*

We follow the same proof outline as in the affine case, with modifications highlighted in gray.

*Proof.* We show equality $I(X_P) = I_{A_P}$ by showing inclusion in both directions.

**Claim**: $I_{A_P} \subseteq I(X_P)$.

We need to show that the binomial generators of $I_{A_P}$ vanish on the variety. Take any $u, v \in \mathbb{N}^{n+1}$ such that $u - v \in \ker(A_P)$. By definition, this means $A_P u = A_P v$. Because the top row of $A_P$ consists entirely of ones, the sum of the coordinates of $u$ equals the sum of the coordinates of $v$, meaning $x^u - x^v$ is a homogeneous binomial. Let $f(x) = x^u - x^v$. We substitute our parameterization into $f$:

$$f(\phi_P(t)) = t^{A_P u} - t^{A_P v} = 0$$

Since the homogeneous polynomial $f$ evaluates to zero for all $t$, the projective variety $V(f)$ contains the entire image $\mathrm{Im}(\phi_P)$. Since the Zariski closure $X_P$ is the smallest variety containing $\mathrm{Im}(\phi_P)$, $V(f)$ must also contain $X_P$. Therefore $f \in I(X_P)$, which proves $I_{A_P} \subseteq I(X_P)$.

**Claim**: $I(X_P) \subseteq I_{A_P}$.

Let $U$ be the set of all homogeneous polynomials that are in $I(X_P)$ but are not in $I_{A_P}$. We want to show that $U$ is empty. Assume $U$ is nonempty. Fix a monomial order on $\mathbb{C}[x_0, \ldots, x_n]$, for example lexicographic order, where $x^a > x^b$ if the first nonzero coordinate of $a - b$ is positive. Since the lexicographic order is a well-ordering, we can pick a polynomial $f \in U$ that has the smallest leading term among all polynomials in $U$. Let this leading term be $cx^u$. Since $f \in I(X_P)$, it must vanish on the parameterization, so we must have

$f(\phi_P(t)) = 0$. When we evaluate $f$, the leading term becomes $ct^{A_P u}$. For our entire polynomial to evaluate to zero, this term $ct^{A_P u}$ must be cancelled out by the evaluation of other terms in $f$, so $f$ must contain at least one other term, say $c'x^v$, such that its evaluation produces a scalar multiple of $t^{A_P v}$ where $A_P u = A_P v$.

Because $A_P u = A_P v$, the difference $u - v$ is in the kernel of $A_P$, so the binomial $x^u - x^v$ is an element of $I_{A_P}$. Since $f$ is a homogeneous polynomial, $x^u$ and $x^v$ have the same degree, so $x^u - x^v$ is a homogeneous binomial of the same degree as $f$. We can then use this binomial to reduce $f$ by defining a new polynomial:

$$g(x) = f(x) - c(x^u - x^v)$$

By subtracting $cx^u$, we have cancelled out the original leading term of $f$. Since $x^v$ was already a monomial appearing in $f$, subtracting it does not introduce any monomials that are lexicographically larger than $x^u$. Because we subtracted a homogeneous polynomial of identical degree, $g(x)$ remains a homogeneous polynomial. Therefore the leading term of $g(x)$ is smaller than the leading term of $f(x)$.

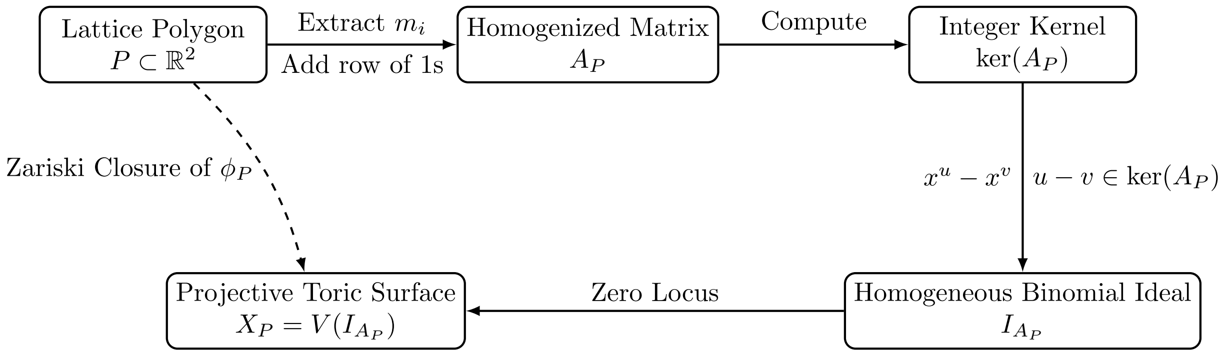
Now is $g$ in the set $U$?

- If $g \in U$, we have found a polynomial in $U$ with a smaller leading term than $f$, contradicting our choice of $f$ as the minimal element.
- If $g \notin U$, it means $g \in I_{A_P}$. But notice $f(x) = g(x) + c(x^u - x^v)$. Since $g \in I_{A_P}$ and $(x^u - x^v) \in I_{A_P}$, their sum $f(x)$ must also be in $I_{A_P}$. Therefore $f \notin U$, which contradicts the fact that we chose $f$ from $U$ to begin with.

Since both cases lead to a contradiction, the set $U$ must be empty. Therefore every homogeneous polynomial in $I(X_P)$ is also in $I_{A_P}$, so $I(X_P) \subseteq I_{A_P}$. This finishes the proof of the projective case. ☐

# 6 Conclusion

The main ideas of this paper are summarized in the following diagram:



By taking a convex lattice polygon $P$, we showed how its integer points give a monomial parameterization that defines a projective toric surface $X_P$ and showed that the implicit defining equations of $X_P$ are generated entirely by homogeneous binomials. Most remarkably, these binomials can be explicitly computed just by finding the kernel of the integer matrix $A_P$ derived from the polygon's vertices. This explicit combinatorial approach to constructing and implicitizing varieties makes toric varieties a powerful tool in combinatorial algebraic geometry.

# References

[1] David A. Cox, John Little, and Donal O'Shea. *Ideals, Varieties, and Algorithms: An Introduction to Computational Algebraic Geometry and Commutative Algebra*. Undergraduate Texts in Mathematics. Springer, 5th edition, 2025.

[2] Jessica Sidman. An introduction to algebraic geometry: Polygons, parameterizations, and equations. *The American Mathematical Monthly*, 119(3):183–198, 2012.