# SPECTRAL EQUIVALENCE OF COMPLEX SOLVABLE LIE ALGEBRAS I: GENERAL THEORY AND HEISENBERG NILRADICALS

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ABSTRACT. We resolve two problems posed by Azari-Key and Yang on the spectral theory of finite-dimensional complex solvable Lie algebras. Using the generalized weights of the nilradical, we show that the spectral invariant $k(\mathfrak{s})$ is determined by the nilradical weight multiset. We also describe the Poincaré polynomial of the eigenvariety complement (with the $z_0$ factor removed) in terms of the Orlik–Solomon algebra of an associated spectral matroid. As an application, we compute explicit spectral invariants for low-dimensional solvable extensions of $\mathfrak{h}(1)$ and $\mathfrak{h}(2)$.

CONTENTS



## 1. INTRODUCTION

1.1. **Solvable Lie Algebras and Nilradicals.** Throughout this paper, we work with finite-dimensional Lie algebras over the complex field $\mathbb{C}$. Classifying these structures is one of the oldest and most important problems in Lie theory. While the classification of semisimple Lie algebras has been essentially solved

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for over a century through the works of Killing and Cartan [Kil88, Car04], the solvable case remains an open problem.

Recall the decomposition theorems of Levi [Lev05] and Malcev [Mal45]. Levi's Theorem states that every finite-dimensional Lie algebra $\mathfrak{g}$ decomposes as a semidirect product $\mathfrak{g} = \mathfrak{l} \ltimes \mathrm{Rad}(\mathfrak{g})$, where $\mathrm{Rad}(\mathfrak{g})$ is the unique maximal solvable ideal and $\mathfrak{l}$ is a semisimple subalgebra. Malcev adds that the Levi factor $\mathfrak{l}$ is unique up to inner automorphism. Together, these results reduce the classification of finite-dimensional Lie algebras to the study of solvable Lie algebras (the radical itself) and the actions of semisimple Lie algebras on them. Of these tasks, the classification of solvable Lie algebras is substantially more challenging. In general, a complete classification cannot be expected. Consequently, the main obstruction to a full classification of finite-dimensional Lie algebras lies in the structure of the solvable radical $\mathrm{Rad}(\mathfrak{g})$.

Since the radical $\mathrm{Rad}(\mathfrak{g})$ is itself a solvable Lie algebra, its structure is closely tied to its maximal nilpotent ideal, the nilradical. Every finite-dimensional solvable Lie algebra $\mathfrak{g}$ over $\mathbb{C}$ contains a unique maximal nilpotent ideal $\mathrm{Nil}(\mathfrak{g})$, and one has $[\mathfrak{g}, \mathfrak{g}] \subseteq \mathrm{Nil}(\mathfrak{g})$. Thus, after choosing a vector-space complement $V_f$ to the nilradical, we may write $\mathfrak{g} = \mathrm{Nil}(\mathfrak{g}) \oplus V_f$. This decomposition is not canonical, and $V_f$ need not be a Lie subalgebra. The adjoint action of elements of $V_f$ on $\mathrm{Nil}(\mathfrak{g})$ gives derivations of the nilradical, well-defined modulo inner derivations after passing to the quotient $\mathfrak{g}/\mathrm{Nil}(\mathfrak{g})$.

This idea motivates the classification strategy initiated by Mubarakzyanov [Mub63c], which proceeds to classify solvable Lie algebras by first fixing a nilpotent Lie algebra and then determining all of the possible extensions that produce a solvable Lie algebra.

Historically, classification efforts have proceeded by dimension. Complete lists of nilpotent Lie algebras are available up to dimension seven [Mor58, See93, Gon98], while computer-assisted approaches have given some results for dimensions eight and nine [TKK00]. In the solvable case, the classification is also complete up to dimension seven. The classification up to dimension six was completed decades ago by [Mub63c, Mub63a, Mub63b, Tur90], with [SW14] providing an exposition of these results. The classification in dimension seven was finished recently by assembling partial results organized according to the dimension of the nilradical: the final remaining cases were those with codimension-one nilradicals [Par13], the four-dimensional case [HT08], and the five-dimensional case [LNN+22].

Following the method introduced by Mubarakzyanov, mathematicians have made substantial progress in classifying solvable Lie algebras with specific nilradicals. In particular, this approach has been successfully applied to solvable Lie algebras with abelian nilradical [NW94], triangular nilradical [TW98], and Borel nilradicals [SW12]. Moreover, there is an extensive literature on solvable Lie algebras with filiform and quasifiliform nilradicals [SW05, SW09, ACSGV06, WLD08, SK10, Sno11, ABCSGV11]. It is also well established that some nilpotent Lie algebras, called characteristically nilpotent, cannot occur as nilradicals of non-nilpotent solvable Lie algebras and should be excluded from the consideration [Kha89, GH94].

This series of papers builds on the spectral approach to finite-dimensional Lie algebras introduced in [HZ19, KY21]. Our goal is to understand the spectral theory of solvable Lie algebras through the structure of their nilradicals. In this paper, we study the general theory and apply it to solvable Lie algebras with Heisenberg nilradical. Subsequent papers will extend this framework to solvable Lie algebras with Borel and triangular nilradicals, and finally to those with filiform and quasifiliform nilradicals.

Among the non-abelian Lie algebras, the Heisenberg Lie algebra $\mathfrak{h}(m)$ stands out as one of the simplest and most important examples of a nilpotent Lie algebra. It arises naturally in several mathematical and physical contexts. In quantum mechanics, it gives the canonical commutation relations between position and momentum operators; in geometry, it appears as the Lie algebra of the Heisenberg group; and in representation theory, it serves as an important example of a non-semisimple Lie algebra with rich representation theory. For a comprehensive treatment of these connections, we refer the interested reader to [Fol89, Kir04, Woi17].

Consequently, a spectral classification of solvable Lie algebras with Heisenberg nilradical is a natural step in the development of spectral theory for finite-dimensional Lie algebras. Physically, these Lie algebras classify the admissible dynamical symmetries of non-linear Schrödinger-type equations [GGRW89] and constitute the solvable subalgebras of the symplectic nuclear collective motion model $\mathfrak{sp}(2n, \mathbb{R}) \ltimes \mathfrak{h}(n)$ [RR77]. In the context of invariant theory, they require the use of rational Casimir operators to distinguish coadjoint orbits where polynomial invariants fail [PSWZ76]. Geometrically, these algebras can be used to construct noncompact Einstein solvmanifolds and Ricci solitons [Heb98, Lau10].

1.2. **Characteristic Polynomials, Spectral Equivalence, and Spectral Invariants.** The spectral theory of a matrix $A \in M_k(\mathbb{C})$ is traditionally developed via its characteristic polynomial $Q_A(\lambda) = \det(\lambda I - A)$. This invariant records the eigenvalues and algebraic multiplicities, and is one of the basic invariants used to study the structure of a matrix. However, algebraic structures arising from physics and geometry are rarely generated by a single matrix, but rather by sets of non-commuting matrices $\{A_1, \dots, A_n\}$. Developing a spectral theory for such sets gives an immediate obstruction, since non-commutativity prevents one from simultaneously diagonalizing the matrices or placing their spectral data into a single polynomial invariant using standard techniques.

The classical forerunner of a multivariate spectral theory is the theory of the group determinant, introduced by Dedekind and Frobenius in the 1890s. Although arising in a purely algebraic context, this construction displays the important features of a multivariate spectral invariant.

Let $G = \{1, g_1, \dots, g_n\}$ be a finite group with representation $\pi$. Recall that the group determinant is defined as:

$$Q_\pi(z) := \det\left(z_0 I + z_1 \pi(g_1) + \cdots + z_n \pi(g_n)\right)$$

Frobenius's breakthrough was the discovery that for the left regular representation, the factorization of this polynomial into irreducible factors completely determines the representation theory of the group.

**Theorem 1.1** (Dedekind-Frobenius, [Goo70, Fro96]). *If $G = \{1, g_1, \dots, g_n\}$ is a finite group and $\lambda_G$ is the left regular representation, then*

$$Q_{\lambda_G}(z) = \prod_{\pi \in \widehat{G}} Q_\pi(z)^{d_\pi},$$

*where $\widehat{G}$ is the set of equivalence classes of irreducible unitary representations of $G$ (called the unitary dual of $G$), and $d_\pi$ is the dimension of the representation $\pi$. Moreover, each such $Q_\pi$ is an irreducible polynomial.*

This theorem marks the start of the study of representation theory for finite groups. For historical developments, see [Cur92, Cur99, Dic21, Dic02, FS91]. Despite this early success, it took more than a century before the group determinant was extended to other algebraic settings. We briefly outline this history; a comprehensive survey is available in [Yan24]. In 2009, [Yan09] extended this notion to unital Banach algebras, initiating a line of research that has since attracted considerable attention [CY13, CSZ15, DY18, GY22, GY17, HWY17, MQW17, SYZ11]. Motivated by these results, [HZ19] defined the characteristic polynomial for finite-dimensional Lie algebras.

**Definition 1.2** (Characteristic Polynomial). Let $L$ be a finite-dimensional Lie algebra with basis $\{x_1, \dots, x_n\}$. The **characteristic polynomial** of $L$ is

$$Q_L(z_0, \mathbf{z}) := \det\left(z_0 I + \sum_{i=1}^{n} z_i \operatorname{ad} x_i\right),$$

where $\operatorname{ad} x_i$ is the adjoint representation of $x_i$ and $\mathbf{z} = (z_1, ..., z_n) \in \mathbb{C}^n$.

This spectral theory has already been applied to study simple [GLW24], classical [FLW26], and solvable Lie algebras [HZ19, Mul23]. The solvable case is particularly interesting: by Lie's theorem, the adjoint operators can be simultaneously upper triangularized over $\mathbb{C}$, and therefore the characteristic polynomial factors completely into linear forms.

**Theorem 1.3** ([HZ19]). *Let $L$ be a solvable Lie algebra over $\mathbb{C}$. Then the characteristic polynomial $Q_L(z_0, \mathbf{z})$ factors as*

$$Q_L(z_0, \mathbf{z}) = \prod_{j=1}^{\dim L} (z_0 + \ell_j(\mathbf{z})),$$

*where each $\ell_j$ is a linear form in $\mathbf{z} = (z_1, \dots, z_n)$.*

One motivation for developing a theory of characteristic polynomials for Lie algebras is the expectation that these polynomials will give spectral invariants capable of distinguishing and classifying Lie algebras. Observe that if two Lie algebras $L_1$ and $L_2$ are isomorphic, any isomorphism corresponds to a change of basis matrix $B \in \mathrm{GL}_n(\mathbb{C})$. A direct calculation shows that their characteristic polynomials must satisfy $Q_{L_2}(z_0, \mathbf{z}) = Q_{L_1}(z_0, \mathbf{z}B)$. Motivated by this observation, [KY21] introduced a coarser classification criterion:

**Definition 1.4** (Spectral Equivalence)**.** Two $n$-dimensional solvable Lie algebras $L_1$ and $L_2$ are **spectrally equivalent** over $\mathbb{C}$, denoted $L_1 \sim L_2$, if their characteristic polynomials satisfy

$$Q_{L_2}(z_0, \mathbf{z}) = Q_{L_1}(z_0, \mathbf{z}B)$$

for some $B \in \mathrm{GL}_n(\mathbb{C})$.

Spectral equivalence is strictly weaker than isomorphism: if $L_1 \cong L_2$, then $L_1 \sim L_2$, but the converse fails in general. However, the converse does hold in certain important cases, such as for semisimple Lie algebras:

**Theorem 1.5** ([Yan24], Corollary 3.29)**.** *Let $L_1$ and $L_2$ be two finite-dimensional complex simple Lie algebras. Then they are isomorphic if and only if $Q_{L_1} = Q_{L_2}$.*

An even stronger result is known in the nilpotent case:

**Theorem 1.6** ([KY21])**.** *A Lie algebra $L$ is nilpotent if and only if its characteristic polynomial is $Q_L(z_0, \mathbf{z}) = z_0^{\dim L}$.*

This does not extend to the broader class of solvable Lie algebras easily. Nevertheless, the factorization from Theorem 1.3 still gives rise to an interesting invariant $k(L)$, introduced in [KY21], which can distinguish some non-isomorphic solvable Lie algebras.

**Definition 1.7** (The Invariant $k(L)$)**.** Let $L$ be a solvable Lie algebra. The integer $k(L)$ is defined as the number of distinct linear factors in the factorization of $Q_L(z_0, \mathbf{z})$.

However, this invariant remains quite mysterious. This motivates the following problem:

**Problem 1.8** ([KY21])**.** Can the spectral invariant $k(L)$ be expressed in terms of other Lie-theoretic invariants?

To date, only partial information about $k(L)$ has been available. The main result currently available is the following proposition due to [KY21].

**Proposition 1.9** ([KY21])**.** *Let $L$ be an $n$-dimensional solvable Lie algebra with basis $\{x_1, x_2, \ldots, x_n\}$. Then*

$$k(L) \geq \max_{1 \leq i \leq n} |\sigma(\operatorname{ad} x_i)|,$$

*where $\sigma(\operatorname{ad} x_i)$ denotes the spectrum of $\operatorname{ad} x_i$ and $|\cdot|$ denotes cardinality.*

Theorem 2.4 establishes that $k(L)$ is completely determined by the nilradical weight multiset; more precisely, it is the number of distinct nilradical weights together with the zero weight. Furthermore, Theorem 2.6 provides sharp lower and upper bounds for this invariant in terms of the dimensions of the extension and the nilradical, respectively.

Additionally, Yang and Azari-Key [KY21] showed that the characteristic polynomial admits several natural geometric interpretations, which in turn give rise to topological spectral invariants.

**Definition 1.10** (Eigenvariety)**.** The **eigenvariety** of $L$ is the zero locus of the characteristic polynomial,

$$V_L := \{(z_0, \mathbf{z}) \in \mathbb{C}^{n+1} : Q_L(z_0, \mathbf{z}) = 0\}.$$

The factor $z_0$ always divides $Q_L(z_0, \mathbf{z})$: at $z_0 = 0$ the matrix in Definition 1.2 is $\operatorname{ad}(\sum_i z_i x_i)$, which has $\sum_i z_i x_i$ in its kernel. Hence the hyperplane $\{z_0 = 0\}$ is present in every eigenvariety. Write $Q_L(z_0, \mathbf{z}) = z_0^a p_L(z_0, \mathbf{z})$, where $a \geq 1$ and $z_0 \nmid p_L$. Define $V_L^* := \{(z_0, \mathbf{z}) \in \mathbb{C}^{n+1} : p_L(z_0, \mathbf{z}) = 0\}$ and $(V_L^*)^c := \mathbb{C}^{n+1} \setminus V_L^*$.

When $L$ is solvable, $p_L$ is a product of homogeneous linear forms, so $V_L^*$ is the union of the hyperplanes defined by the remaining factors of $Q_L$ after the factor $z_0$ has been removed. Throughout this paper, the Betti numbers and Poincaré polynomial are attached to the eigenvariety complement $(V_L^*)^c$, rather than to $\mathbb{C}^{n+1} \setminus V_L$. Equivalently, the component $\{z_0 = 0\}$ is removed before the complement is taken.

**Definition 1.11** (Poincaré polynomial)**.** Let $L$ be a finite-dimensional Lie algebra, and write $Q_L(z_0, \mathbf{z}) = z_0^a p_L(z_0, \mathbf{z})$ with $a \geq 1$ and $z_0 \nmid p_L$. Let $V_L^* = \{p_L = 0\} \subset \mathbb{C}^{n+1}$. For each $j \geq 0$, the $j$-th **Betti number** of $L$ is $b_j(L) := \dim_{\mathbb{C}} H^j((V_L^*)^c, \mathbb{C})$. The **Poincaré polynomial** of $L$ is $P_L(t) := \sum_{j\geq 0} b_j(L)t^j$.

Yang and Azari-Key also proved two additional results:

**Proposition 1.12** ([KY21])**.** *If $L$ is a solvable Lie algebra, then $b_1(L) = k(L) - 1$.*

**Proposition 1.13** ([KY21])**.** *For any finite-dimensional solvable Lie algebra $L$,*

$$\deg P_L \leq \dim\bigl(L/\operatorname{Nil}(L)\bigr) + 1.$$

However, the structure of the higher Betti numbers is still an open problem:

**Problem 1.14** ([KY21])**.** Can the Betti numbers $b_j(L)$, $j \geq 2$ be determined by known invariants of $L$?

We resolve this problem in Section 2.2. With the convention above, the relevant complement is $(V_{\mathfrak{s}}^*)^c$, where the $z_0$ factor has been removed from $Q_{\mathfrak{s}}$. We identify its cohomology with the Orlik–Solomon algebra of a matroid. Theorem 2.20 then shows that the Poincaré polynomial is determined by the Whitney numbers of the first kind: $b_j(\mathfrak{s}) = w_j$. Theorem 2.21 determines the degree of the Poincaré polynomial as the rank of the arrangement defined by the factors $z_0 + \alpha$, where $\alpha$ ranges over the nonzero nilradical weights. Finally, Corollary 2.27 establishes log-concavity of the resulting Betti numbers.

Finally, in Section 3, we apply this general framework to solvable Lie algebras with Heisenberg nilradical. We derive a canonical factorization of the characteristic polynomial that explicitly isolates the symplectic structure of the nilradical (Theorem 3.3). This factorization reveals constraints on the topology of $(V_{\mathfrak{s}}^*)^c$. In particular, Theorem 3.7 implies that the Betti numbers $b_j(\mathfrak{s})$ vanish for $j > m+2$. Finally, we compute explicit spectral invariants for low-dimensional solvable extensions of $\mathfrak{h}(1)$ and $\mathfrak{h}(2)$.

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## 2. General: Over $\mathbb{C}$

### 2.1. The Spectral Invariant $k(L)$.

Let $\mathfrak{s}$ be a finite-dimensional solvable Lie algebra over $\mathbb{C}$ with nilradical $\mathfrak{n}$. Choose a vector-space complement $V_f$ so that $\mathfrak{s} = \mathfrak{n} \oplus V_f$, and write $k = \dim V_f$. Since $\mathfrak{n}$ is an ideal, the adjoint action restricts to a representation $\rho : \mathfrak{s} \to \mathfrak{gl}(\mathfrak{n})$, where $\rho(y) = \operatorname{ad}(y)|_{\mathfrak{n}}$.

By Lie's theorem, there is a basis of $\mathfrak{n}$ in which every $\rho(y)$ is upper triangular. Thus there are linear functionals $\lambda_1, \ldots, \lambda_M \in \mathfrak{s}^*$, where $M = \dim \mathfrak{n}$, such that the diagonal entries of $\rho(y)$ are $\lambda_1(y), \ldots, \lambda_M(y)$ for every $y \in \mathfrak{s}$. If $x \in \mathfrak{n}$, then $\rho(x)$ is nilpotent, so every $\lambda_i$ vanishes on $\mathfrak{n}$. Hence each $\lambda_i$ descends to $(\mathfrak{s}/\mathfrak{n})^*$, and after choosing $V_f$, we regard it as an element of $V_f^*$.

**Definition 2.1.** The **nilradical weight multiset** of $\mathfrak{s}$ is $\Delta_{\mathfrak{s}} := \{\lambda_1|_{V_f}, \ldots, \lambda_M|_{V_f}\}$, counted with multiplicity. We write $\Delta$ for the corresponding set of distinct weights. If $\alpha \in \Delta$, let $m_\alpha$ denote the multiplicity of $\alpha$.

**Lemma 2.2.** *The nilradical weight multiset $\Delta_{\mathfrak{s}}$ is independent of the choice of triangularizing basis. Equivalently, it is the multiset of weights with which $\mathfrak{s}$ acts on the one-dimensional quotients in any composition series of the $\mathfrak{s}$-module $\mathfrak{n}$. Each of these weights vanishes on $\mathfrak{n}$, so $\Delta_{\mathfrak{s}}$ is naturally a multiset in $(\mathfrak{s}/\mathfrak{n})^*$. After choosing a complement $V_f$ to $\mathfrak{n}$, we identify $(\mathfrak{s}/\mathfrak{n})^*$ with $V_f^*$.*

*Proof.* Let $\rho : \mathfrak{s} \to \mathfrak{gl}(\mathfrak{n})$ be given by $\rho(y) = \operatorname{ad}(y)|_{\mathfrak{n}}$. By Lie's theorem, $\mathfrak{n}$ has a complete flag of $\mathfrak{s}$-submodules $0 = N_0 \subset N_1 \subset \cdots \subset N_M = \mathfrak{n}$, with $\dim N_i/N_{i-1} = 1$. For each $i$, the action of $\mathfrak{s}$ on

$N_i/N_{i-1}$ is given by a linear functional $\lambda_i \in \mathfrak{s}^*$. These functionals are the weights appearing on the diagonal in any simultaneous upper triangular form of the operators $\rho(y)$.

The multiset $\{\lambda_1, \dots, \lambda_M\}$ is independent of the chosen flag, since it is the multiset of composition factors of the finite-dimensional $\mathfrak{s}$-module $\mathfrak{n}$, counted with multiplicity. If $x \in \mathfrak{n}$, then $\rho(x) = \mathrm{ad}(x)|_{\mathfrak{n}}$ is nilpotent because $\mathfrak{n}$ is nilpotent. Hence $x$ acts by zero on every one-dimensional quotient $N_i/N_{i-1}$, so $\lambda_i(x) = 0$ for every $i$. Thus each $\lambda_i$ vanishes on $\mathfrak{n}$ and descends to a functional on $\mathfrak{s}/\mathfrak{n}$. Choosing $V_f$ identifies this quotient dual with $V_f^*$. □

*Remark* 2.3. The weights in Definition 2.1 are generalized weights. No simultaneous diagonalization is assumed. In particular, we do not require the Jordan–Chevalley semisimple parts of the operators $\mathrm{ad}(f)|_{\mathfrak{n}}$ to commute.

Choose a basis $\mathcal{B} = (x_1, \dots, x_M, f_1, \dots, f_k)$ of $\mathfrak{s}$ adapted to the decomposition $\mathfrak{s} = \mathfrak{n} \oplus V_f$, where $M = \dim \mathfrak{n}$ and $k = \dim V_f$. Let $N = M + k$ denote the total dimension of $\mathfrak{s}$. We associate this basis with the set of indeterminates $\mathbf{z} = (z_{x_1}, \dots, z_{x_M}, z_{f_1}, \dots, z_{f_k})$. For each weight $\alpha \in \Delta$, define the linear form

$$\ell_\alpha(z_f) := \sum_{j=1}^{k} \alpha(f_j)\, z_{f_j}.$$

Recall that the characteristic polynomial is defined as the determinant of the $N \times N$ matrix pencil $\mathbf{A}(z_0, \mathbf{z})$:

$$\mathbf{A}(z_0, \mathbf{z}) := z_0 I_N + \sum_{i=1}^{M} z_{x_i}\, \mathrm{ad}(x_i) + \sum_{j=1}^{k} z_{f_j}\, \mathrm{ad}(f_j),$$

where $I_N$ is the identity matrix of dimension $N \times N$, and the operators $\mathrm{ad}(x_i), \mathrm{ad}(f_j)$ are the $N \times N$ adjoint matrices of the basis elements.

With the weight structure established, we use it to study the characteristic polynomial. Our approach is guided by the goal of obtaining a statement analogous to the Dedekind-Frobenius Theorem (Theorem 1.1), where the factorization of the group determinant is indexed by the unitary dual of a finite group. In the Lie algebra setting, it is natural to ask whether the characteristic polynomial $Q_{\mathfrak{s}}(z)$ admits a similar structural interpretation.

For finite-dimensional simple Lie algebras, this question has been addressed through the lens of invariant theory and Weyl group actions. Feng, Liu, and Wang [FLW26] showed that for classical Lie algebras and the exceptional algebra $G_2$, the characteristic polynomial associated with a finite-dimensional representation decomposes into a product of irreducible orbital factors. These factors are uniquely determined by the orbits of the weights of the representation under the action of the corresponding Weyl group.

The theorem below shows that for solvable Lie algebras, the generalized weights of the action on the nilradical index the linear factors of $Q_{\mathfrak{s}}(z)$, with multiplicities given by the multiplicities of the corresponding weights in any simultaneous triangularization.

This result also resolves Problem 1.8, posed by [KY21], which asks whether the spectral invariant $k(\mathfrak{s})$ can be expressed in terms of other well-known Lie-theoretic invariants.

**Theorem 2.4.** *Let $\mathfrak{s}$ be a finite-dimensional complex solvable Lie algebra with nilradical $\mathfrak{n}$, and choose a vector space complement $V_f$ so that $\mathfrak{s} = \mathfrak{n} \oplus V_f$. Let $\Delta \subset V_f^*$ be the set of distinct weights from Definition 2.1, and let $m_\alpha$ be the multiplicity of $\alpha$ in the nilradical weight multiset. Then the characteristic polynomial of $\mathfrak{s}$ factors as*

$$Q_{\mathfrak{s}}(z) = z_0^{\dim V_f} \prod_{\alpha \in \Delta} \big(z_0 + \ell_\alpha(z_f)\big)^{m_\alpha}.$$

*Consequently, the spectral invariant is $k(\mathfrak{s}) = |\Delta \cup \{0\}|$.*

*Proof.* With respect to the decomposition $\mathfrak{s} = \mathfrak{n} \oplus V_f$, the matrix pencil $\mathbf{A}(z_0, \mathbf{z})$ is block upper-triangular:

$$\mathbf{A}(z_0, \mathbf{z}) = \begin{pmatrix} \mathbf{A}_{\mathfrak{n}}(z_0, \mathbf{z}) & * \\ 0 & \mathbf{A}_{\mathfrak{s}/\mathfrak{n}}(z_0, \mathbf{z}) \end{pmatrix}.$$

Here $\mathbf{A}_{\mathfrak{n}}$ is the restriction of the pencil to the invariant subspace $\mathfrak{n}$, while $\mathbf{A}_{\mathfrak{s}/\mathfrak{n}}$ denotes the induced pencil on the quotient $\mathfrak{s}/\mathfrak{n}$, identified with $V_f$ as a vector space. Taking determinants gives

$$Q_{\mathfrak{s}}(z) = \det(\mathbf{A}_{\mathfrak{n}}(z_0, \mathbf{z})) \det(\mathbf{A}_{\mathfrak{s}/\mathfrak{n}}(z_0, \mathbf{z})).$$

We first compute the quotient block. Since $\mathfrak{s}$ is solvable, Lie's theorem gives a basis in which all operators $\mathrm{ad}(\mathfrak{s})$ are upper triangular. Hence all operators $\mathrm{ad}([\mathfrak{s}, \mathfrak{s}])$ are strictly upper triangular, and therefore nilpotent. By Engel's theorem, $[\mathfrak{s}, \mathfrak{s}]$ is a nilpotent ideal, so by maximality of the nilradical,

$$[\mathfrak{s}, \mathfrak{s}] \subseteq \mathfrak{n}.$$

Thus the induced adjoint action of $\mathfrak{s}$ on $\mathfrak{s}/\mathfrak{n}$ is trivial. Hence

$$\mathbf{A}_{\mathfrak{s}/\mathfrak{n}}(z_0, \mathbf{z}) = z_0 I_{\dim V_f}, \qquad \det(\mathbf{A}_{\mathfrak{s}/\mathfrak{n}}(z_0, \mathbf{z})) = z_0^{\dim V_f}.$$

Next, consider $\det(\mathbf{A}(z_0, \mathbf{z})|_{\mathfrak{n}})$. Choose a basis of $\mathfrak{n}$ in which all operators $\rho(y) = \mathrm{ad}(y)|_{\mathfrak{n}}$, $y \in \mathfrak{s}$, are upper triangular. Let $\lambda_1, \ldots, \lambda_M$ be the corresponding weights. Since each $\lambda_i$ vanishes on $\mathfrak{n}$, the variables $z_{x_1}, \ldots, z_{x_M}$ do not appear on the diagonal of the nilradical block. If $\lambda_i|_{V_f} = \alpha$, then the corresponding diagonal entry is $z_0 + \ell_\alpha(z_f)$. Hence the nilradical block contributes $\prod_{\alpha \in \Delta}(z_0 + \ell_\alpha(z_f))^{m_\alpha}$.

Combining this with the quotient block gives

$$Q_{\mathfrak{s}}(z) = z_0^{\dim V_f} \prod_{\alpha \in \Delta} (z_0 + \ell_\alpha(z_f))^{m_\alpha}.$$

The distinct linear factors are $z_0$ together with the factors $z_0 + \ell_\alpha(z_f)$ for those $\alpha \in \Delta$ with $\alpha \neq 0$. Equivalently, the distinct factors are indexed by $\Delta \cup \{0\}$, where the element 0 corresponds to the factor $z_0$. Hence

$$k(\mathfrak{s}) = |\Delta \cup \{0\}|.$$

□

*Remark* 2.5. For solvable Lie algebras, Theorem 2.4 recovers the nilpotence criterion from [KY21, Mul23]. If $L$ is nilpotent, then $L = \mathfrak{n}$ and $Q_L(z) = z_0^{\dim L}$. Conversely, if $Q_L(z) = z_0^{\dim L}$, then every nilradical weight is zero. Hence every $f \in V_f$ acts nilpotently on $\mathfrak{n}$, and the induced action on $L/\mathfrak{n}$ is trivial. Thus $\mathrm{ad}(f)$ is nilpotent on $L$. Since, for a finite-dimensional solvable Lie algebra over $\mathbb{C}$, the nilradical is the set of ad-nilpotent elements, we get $f \in \mathfrak{n}$. Therefore $V_f = 0$ and $L = \mathfrak{n}$ is nilpotent.

We can use this theorem to obtain sharp bounds for the spectral invariant $k(\mathfrak{s})$.

**Theorem 2.6.** *Let $\mathfrak{s}$ be a solvable Lie algebra with nilradical $\mathfrak{n}$ and chosen complement $V_f$. Then the spectral invariant satisfies*

$$\dim V_f + 1 \leq k(\mathfrak{s}) \leq \dim \mathfrak{n} + 1.$$

*Moreover, $k(\mathfrak{s}) = \dim V_f + 1$ if and only if the nonzero weights $\Delta \setminus \{0\}$ form a basis of $V_f^*$. Also, $k(\mathfrak{s}) = \dim \mathfrak{n} + 1$ if and only if $0 \notin \Delta$ and every distinct weight has multiplicity one, that is, $m_\alpha = 1$ for all $\alpha \in \Delta$.*

*Proof.* We prove the two bounds separately:

- Lower Bound: We first show that the set of weights spans the dual space. Suppose $\mathrm{span}(\Delta) \subsetneq V_f^*$. Then there exists a nonzero $f_0 \in V_f$ such that $\alpha(f_0) = 0$ for all $\alpha \in \Delta$. The eigenvalues of $\mathrm{ad}(f_0)|_{\mathfrak{n}}$ are $\{\alpha(f_0) \mid \alpha \in \Delta\} = \{0\}$, so this operator is nilpotent. Moreover, as shown in the proof of Theorem 2.4, the induced action on $\mathfrak{s}/\mathfrak{n}$ is trivial, hence nilpotent. Therefore $\mathrm{ad}(f_0)$ is globally nilpotent on $\mathfrak{s}$, so $f_0$ belongs to the nilradical $\mathfrak{n}$. This contradicts $f_0 \in V_f \setminus \{0\}$.

  Thus $\mathrm{span}(\Delta) = V_f^*$. Since the zero vector does not contribute to the span, the set of non-zero weights $\Delta' := \Delta \setminus \{0\}$ must span $V_f^*$. Consequently, $|\Delta'| \geq \dim(V_f)$.

  Recall that $k(\mathfrak{s})$ counts the number of distinct linear factors in $Q_{\mathfrak{s}}(z)$. By Theorem 2.4, the extension $V_f$ contributes the factor $z_0^{\dim V_f}$. Thus the factor $z_0$ is always present in $Q_{\mathfrak{s}}$, coming either from the quotient block or, if $0 \in \Delta$, also from zero nilradical weights. Hence the distinct factors are indexed by $\Delta \cup \{0\}$. We can compute the cardinality as:

$$k(\mathfrak{s}) = |\Delta \cup \{0\}| = |\Delta'| + 1.$$

Combining this with the span inequality $|\Delta'| \geq \dim(V_f)$, we obtain:

$$k(\mathfrak{s}) \geq \dim(V_f) + 1.$$

Equality holds if and only if $|\Delta'| = \dim(V_f)$, which occurs if and only if the non-zero weights $\Delta'$ form a basis of $V_f^*$.

- Upper Bound: Since the nilradical weight multiset has total cardinality $\dim \mathfrak{n}$, we have $\sum_{\alpha \in \Delta} m_\alpha = \dim \mathfrak{n}$. In particular, $|\Delta| \leq \dim \mathfrak{n}$. If $0 \in \Delta$, then $k(\mathfrak{s}) = |\Delta| \leq \dim \mathfrak{n}$. If $0 \notin \Delta$, then $k(\mathfrak{s}) = |\Delta| + 1 \leq \dim \mathfrak{n} + 1$. This proves the bound.

  Equality requires $0 \notin \Delta$ and $|\Delta| = \dim \mathfrak{n}$. Since $\sum_{\alpha \in \Delta} m_\alpha = \dim \mathfrak{n}$ and every $m_\alpha \geq 1$, this happens if and only if $m_\alpha = 1$ for every $\alpha \in \Delta$.

□

The factorization of Theorem 2.4 allows the notion of spectral equivalence to be reformulated as a condition on the collection of weights.

**Proposition 2.7.** *Let $\mathfrak{s}$ and $\mathfrak{s}'$ be $N$-dimensional complex solvable Lie algebras. Let $\Delta_{\mathfrak{s}}^\times$ and $\Delta_{\mathfrak{s}'}^\times$ be the multisets of nonzero nilradical weights, and let $e_0(\mathfrak{s})$ and $e_0(\mathfrak{s}')$ be the total multiplicities of the factor $z_0$ in $Q_{\mathfrak{s}}$ and $Q_{\mathfrak{s}'}$. Then $\mathfrak{s} \sim \mathfrak{s}'$ if and only if $e_0(\mathfrak{s}) = e_0(\mathfrak{s}')$ and there is a linear isomorphism $\operatorname{span} \Delta_{\mathfrak{s}}^\times \to \operatorname{span} \Delta_{\mathfrak{s}'}^\times$ carrying $\Delta_{\mathfrak{s}}^\times$ to $\Delta_{\mathfrak{s}'}^\times$ as multisets.*

*Proof.* By Theorem 2.4, the characteristic polynomial of $\mathfrak{s}$ has the form

$$Q_{\mathfrak{s}} = z_0^{e_0(\mathfrak{s})} \prod_{\alpha \in \Delta_{\mathfrak{s}}^\times} (z_0 + \ell_\alpha),$$

where the product is over the nonzero nilradical weight multiset, counted with multiplicity. Here each $\ell_\alpha$ is viewed as a linear form on the full space of Lie-algebra variables by extending it by zero on the nilradical variables. The same description holds for $\mathfrak{s}'$.

Suppose first that $\mathfrak{s} \sim \mathfrak{s}'$. Then $Q_{\mathfrak{s}'}(z_0, \mathbf{z}) = Q_{\mathfrak{s}}(z_0, \mathbf{z}B)$ for some $B \in \mathrm{GL}_N(\mathbb{C})$. This change of variables fixes $z_0$. Therefore the irreducible factor $z_0$ is carried to $z_0$, and unique factorization gives $e_0(\mathfrak{s}) = e_0(\mathfrak{s}')$. The remaining factors are the factors $z_0 + \ell_\alpha$ with $\alpha \neq 0$. Since an invertible linear change of the Lie-algebra variables cannot send a nonzero linear form to zero, it carries the nonzero factors of $Q_{\mathfrak{s}}$ to the nonzero factors of $Q_{\mathfrak{s}'}$, with multiplicity. After subtracting the common $z_0$ term, we obtain a linear isomorphism from $\operatorname{span} \Delta_{\mathfrak{s}}^\times$ to $\operatorname{span} \Delta_{\mathfrak{s}'}^\times$ carrying one multiset to the other.

Conversely, suppose that $e_0(\mathfrak{s}) = e_0(\mathfrak{s}')$ and that such a linear isomorphism $T : \operatorname{span} \Delta_{\mathfrak{s}}^\times \to \operatorname{span} \Delta_{\mathfrak{s}'}^\times$ exists. Extend $T$ to an isomorphism between the full dual spaces of Lie-algebra variables by choosing complements, and let the corresponding change of variables fix $z_0$. Then each factor $z_0 + \ell_\alpha$ is carried to the corresponding factor for $\mathfrak{s}'$, with the same multiplicity, and the exponent of $z_0$ also agrees. Hence the characteristic polynomials are related by an invertible change of variables, so $\mathfrak{s} \sim \mathfrak{s}'$. □

*Remark* 2.8. If one works inside a family where $\dim V_f$ and the multiplicity of the zero nilradical weight are fixed, this is equivalent to the simpler condition that the full nilradical weight multisets are equivalent under $\mathrm{GL}(V_f^*)$.

Many classification lists contain families of solvable Lie algebras depending on continuous parameters. Let $\mathfrak{s}(\mathbf{p})$ be such a family of $N$-dimensional complex solvable Lie algebras, with parameter $\mathbf{p} \in \mathcal{P}$. Since all nilpotent Lie algebras of a fixed dimension have characteristic polynomial $z_0^N$, spectral equivalence is very coarse on nilpotent algebras. For non-nilpotent solvable algebras, however, the nonzero nilradical weights can vary with the parameters. This motivates the following definition.

**Definition 2.9.** A parameterized family of Lie algebras $\mathfrak{s}(\mathbf{p})$ is **spectrally rigid** if for any $\mathbf{p}, \mathbf{p}' \in \mathcal{P}$, the condition $\mathfrak{s}(\mathbf{p}) \sim \mathfrak{s}(\mathbf{p}')$ implies $\mathbf{p} = \mathbf{p}'$.

By Proposition 2.7, spectral equivalence is controlled by the total multiplicity of the factor $z_0$ and by the orbit of the nonzero nilradical weight multiset. Thus, in any family where $e_0(\mathfrak{s}(\mathbf{p}))$ is constant and the nonzero weights span a fixed vector space $V_f^*$, spectral rigidity is equivalent to separation of the nonzero weight multisets under $\mathrm{GL}(V_f^*)$.

**Corollary 2.10.** *Let $\mathfrak{s}(\mathbf{p})$ be a family of solvable Lie algebras for which $e_0(\mathfrak{s}(\mathbf{p}))$ is constant and all nonzero weight multisets are viewed in a fixed space $V_f^*$. Let $\Delta^\times(\mathbf{p})$ denote the nonzero nilradical weight multiset. Then the family is spectrally rigid if and only if, for every $\mathbf{p} \neq \mathbf{p}'$, there is no $T \in \mathrm{GL}(V_f^*)$ such that $T(\Delta^\times(\mathbf{p})) = \Delta^\times(\mathbf{p}')$ as multisets.*

2.2. **Topological Spectral Invariants.** We now study the topological consequences of Theorem 2.4. Write $Q_\mathfrak{s} = z_0^a p_\mathfrak{s}$, where $a = e_0(\mathfrak{s})$ and $z_0 \nmid p_\mathfrak{s}$. The complement is $(V_\mathfrak{s}^*)^c = \mathbb{C}^{N+1} \setminus V_\mathfrak{s}^*$, where $V_\mathfrak{s}^* = \{p_\mathfrak{s} = 0\}$. Let $\Delta^\times$ be the set of distinct nonzero nilradical weights. Then the distinct linear factors of $p_\mathfrak{s}$ are $z_0 + \ell_\alpha$ for $\alpha \in \Delta^\times$. Set $\Phi_\mathfrak{s}^* := \{z_0 + \ell_\alpha : \alpha \in \Delta^\times\} \subset (\mathbb{C} \oplus V_f)^*$, and let $\mathcal{A}_\mathfrak{s}^* := \{\ker L : L \in \Phi_\mathfrak{s}^*\}$.

Since $p_\mathfrak{s}$ depends only on $z_0$ and the variables attached to $V_f$, the nilradical variables give a contractible affine factor. Thus

$$(V_\mathfrak{s}^*)^c \cong \mathbb{C}^{\dim \mathfrak{n}} \times M(\mathcal{A}_\mathfrak{s}^*),$$

where

$$M(\mathcal{A}_\mathfrak{s}^*) = (\mathbb{C} \oplus V_f) \setminus \bigcup_{H \in \mathcal{A}_\mathfrak{s}^*} H.$$

Therefore the cohomology of $(V_\mathfrak{s}^*)^c$ is the same as the cohomology of the arrangement complement $M(\mathcal{A}_\mathfrak{s}^*)$.

For each $\ell \in \Phi_\mathfrak{s}^*$, set $\omega_\ell = (2\pi i)^{-1} d\ell/\ell$. Let $\mathfrak{A}(\mathcal{A}_\mathfrak{s}^*)$ be the graded $\mathbb{C}$-subalgebra of differential forms on $M(\mathcal{A}_\mathfrak{s}^*)$ generated by 1 and the forms $\omega_\ell$ for $\ell \in \Phi_\mathfrak{s}^*$. We write $[\omega_\ell]$ for the corresponding de Rham cohomology class.

Orlik and Solomon [OS80] determined this cohomology algebra entirely from the linear dependence relations among the defining linear forms. We recall the standard form of their theorem.

**Theorem 2.11** ([OS80], Theorem 5.2)**.** *Let $\mathcal{A}$ be a finite complex hyperplane arrangement with complement $M(\mathcal{A})$. Let $A(\mathcal{A}) = E/I$ be its Orlik–Solomon algebra, where $E$ is the exterior algebra generated by symbols $e_H$ indexed by $H \in \mathcal{A}$, and $I$ is the Orlik–Solomon ideal generated by the boundaries of dependent sets. Then there is an isomorphism of graded algebras*

$$A(\mathcal{A}) \cong H^*(M(\mathcal{A}), \mathbb{C}),$$

*sending $e_H$ to the cohomology class of $(2\pi i)^{-1} d\ell_H/\ell_H$, where $\ell_H$ is any defining linear form for $H$.*

Applying this theorem to the arrangement $\mathcal{A}_\mathfrak{s}^*$ gives the following presentation of the cohomology ring.

**Corollary 2.12.** *Let $\mathcal{A}_\mathfrak{s}^*$ be the hyperplane arrangement defined by the distinct linear factors of $p_\mathfrak{s}(z_0, \mathbf{z})$, and let $E$ be the exterior algebra generated by $\{e_\ell\}_{\ell \in \Phi_\mathfrak{s}^*}$. Then*

$$H^*((V_\mathfrak{s}^*)^c, \mathbb{C}) \cong H^*(M(\mathcal{A}_\mathfrak{s}^*), \mathbb{C}) \cong E/I,$$

*where $I$ is the Orlik–Solomon ideal generated by $\partial(e_S)$ for all dependent subsets $S \subseteq \Phi_\mathfrak{s}^*$.*

*Proof.* The polynomial $p_\mathfrak{s}$ depends only on $z_0$ and the variables attached to $V_f$. Hence $(V_\mathfrak{s}^*)^c \cong \mathbb{C}^{\dim \mathfrak{n}} \times M(\mathcal{A}_\mathfrak{s}^*)$. Since $\mathbb{C}^{\dim \mathfrak{n}}$ is contractible, projection onto the second factor induces an isomorphism in cohomology. The second isomorphism is the Orlik–Solomon theorem applied to $\mathcal{A}_\mathfrak{s}^*$. □

This corollary reduces the computation of the cohomology ring to the combinatorics of linear dependencies, which can be studied through the theory of matroids.

**Definition 2.13** (Spectral Matroid)**.** Let $\Phi_\mathfrak{s}^*$ be the set of distinct linear factors of $p_\mathfrak{s}$, where $Q_\mathfrak{s} = z_0^{e_0(\mathfrak{s})} p_\mathfrak{s}$ and $z_0 \nmid p_\mathfrak{s}$. The **spectral matroid** $\mathcal{M}_\mathfrak{s}^*$ is the linear matroid represented by $\Phi_\mathfrak{s}^* \subset (\mathbb{C} \oplus V_f)^*$.

A useful property of the spectral matroid is that it is an invariant of the spectral equivalence class.

**Lemma 2.14.** *The spectral matroid $\mathcal{M}_\mathfrak{s}^*$ is a well-defined linear matroid. Furthermore, if $\mathfrak{s} \sim \mathfrak{s}'$, then $\mathcal{M}_\mathfrak{s}^* \cong \mathcal{M}_{\mathfrak{s}'}^*$.*

*Proof.* The first claim follows immediately from the definition of a linear matroid represented by the set of linear forms $\Phi_\mathfrak{s}^*$. Suppose $\mathfrak{s} \sim \mathfrak{s}'$. By Proposition 2.7, the total $z_0$-multiplicities agree and there is a linear isomorphism $T$ between the spans of the nonzero nilradical weight multisets carrying $\Delta_{\mathfrak{s}}^{\times}$ to $\Delta_{\mathfrak{s}'}^{\times}$ as multisets.

The ground set $\Phi_{\mathfrak{s}}^*$ consists of the forms $z_0 + \alpha$, where $\alpha$ ranges over the distinct nonzero nilradical weights. The map $z_0 + \alpha \mapsto z_0 + T(\alpha)$ gives a bijection $\Phi_{\mathfrak{s}}^* \to \Phi_{\mathfrak{s}'}^*$. A subset $\{z_0 + \alpha_1, \ldots, z_0 + \alpha_r\}$ is linearly dependent if and only if there are coefficients $c_1, \ldots, c_r$, not all zero, such that $\sum_i c_i = 0$ and $\sum_i c_i\alpha_i = 0$. Since $T$ is a linear isomorphism on the span of the weights, this condition is equivalent to $\sum_i c_i = 0$ and $\sum_i c_i T(\alpha_i) = 0$. Thus the bijection preserves the dependent subsets, and therefore induces an isomorphism $\mathcal{M}_{\mathfrak{s}}^* \cong \mathcal{M}_{\mathfrak{s}'}^*$. ☐

**Proposition 2.15.** *For any finite-dimensional solvable Lie algebra $\mathfrak{s}$, the cohomology ring $H^*((V_{\mathfrak{s}}^*)^c, \mathbb{C})$ is a spectral invariant determined by the spectral matroid $\mathcal{M}_{\mathfrak{s}}^*$.*

*Proof.* By construction, $(V_{\mathfrak{s}}^*)^c$ is isomorphic to $\mathbb{C}^{\dim \mathfrak{n}} \times M(\mathcal{A}_{\mathfrak{s}}^*)$, where $\mathcal{A}_{\mathfrak{s}}^*$ is the arrangement represented by the set $\Phi_{\mathfrak{s}}^*$ of distinct linear factors of $p_{\mathfrak{s}}$. Since $\mathbb{C}^{\dim \mathfrak{n}}$ is contractible, $(V_{\mathfrak{s}}^*)^c$ and $M(\mathcal{A}_{\mathfrak{s}}^*)$ have the same cohomology. By Corollary 2.12, the cohomology ring $H^*((V_{\mathfrak{s}}^*)^c, \mathbb{C})$ is isomorphic to the Orlik–Solomon algebra of the matroid represented by $\Phi_{\mathfrak{s}}^*$, namely $\mathcal{M}_{\mathfrak{s}}^*$.

If $\mathfrak{s} \sim \mathfrak{s}'$, then Lemma 2.14 gives an isomorphism $\mathcal{M}_{\mathfrak{s}}^* \cong \mathcal{M}_{\mathfrak{s}'}^*$. The Orlik–Solomon algebra is determined by the dependent subsets of the matroid, so isomorphic matroids have isomorphic Orlik–Solomon algebras. Therefore $H^*((V_{\mathfrak{s}}^*)^c, \mathbb{C})$ is determined by $\mathcal{M}_{\mathfrak{s}}^*$ and is invariant under spectral equivalence. ☐

To compute the Betti numbers of $(V_{\mathfrak{s}}^*)^c$ explicitly, we introduce the tools related to the intersection lattice.

**Definition 2.16** (Intersection Lattice)**.** Let $\mathcal{A}_{\mathfrak{s}}^*$ be the hyperplane arrangement in $\mathbb{C} \oplus V_f$ defined by the distinct linear factors of $p_{\mathfrak{s}}(z_0, \mathbf{z})$. The **intersection lattice** $L(\mathcal{A}_{\mathfrak{s}}^*)$ consists of all subspaces $X \subseteq \mathbb{C} \oplus V_f$ of the form

$$X = \bigcap_{H \in S} H$$

for some $S \subseteq \mathcal{A}_{\mathfrak{s}}^*$. The partial order is given by reverse inclusion: $X \leq Y$ if $Y \subseteq X$. The minimal element is $\hat{0} = \mathbb{C} \oplus V_f$, and the maximal element is

$$\hat{1} = \bigcap_{H \in \mathcal{A}_{\mathfrak{s}}^*} H.$$

The rank function is $\operatorname{rank}(X) = \operatorname{codim}_{\mathbb{C} \oplus V_f}(X)$.

The inclusion–exclusion structure of the arrangement is given by the Möbius function of this lattice.

**Definition 2.17** (Möbius Function)**.** The **Möbius function** $\mu\colon L(\mathcal{A}_{\mathfrak{s}}^*) \times L(\mathcal{A}_{\mathfrak{s}}^*) \to \mathbb{Z}$ is characterized by $\mu(X, X) = 1$ for all $X$, and by the relation

$$\sum_{X \leq Z \leq Y} \mu(X, Z) = 0 \quad \text{for } X < Y.$$

We write $\mu(X) = \mu(\hat{0}, X)$.

To express the Betti numbers combinatorially, we introduce the Whitney numbers of the first kind.

**Definition 2.18** (Whitney Numbers)**.** The **Whitney numbers of the first kind** are defined by

$$w_k = \sum_{\operatorname{rank}(X) = k} |\mu(X)|.$$

Since $\mathcal{A}_{\mathfrak{s}}^*$ is a central complex hyperplane arrangement, its intersection lattice is a finite geometric lattice, with rank given by codimension in $\mathbb{C} \oplus V_f$. We use the following standard form of the Orlik–Solomon formula.

**Theorem 2.19** ([OS80], Theorem 2.6)**.** *Let $L$ be a finite geometric lattice. The Poincaré polynomial of the associated graded Orlik–Solomon algebra is*

$$P(t) = \sum_{X \in L} \mu(X)(-t)^{\operatorname{rank}(X)},$$

*where $\mu(X) = \mu(\hat{0}, X)$.*

Applying this to $L(\mathcal{A}_{\mathfrak{s}}^*)$ gives the Poincaré polynomial of the eigenvariety complement.

**Theorem 2.20.** *Let $\mathcal{A}_{\mathfrak{s}}^*$ be the central arrangement defined by the distinct linear factors of $p_{\mathfrak{s}}(z_0, \mathbf{z})$, where $Q_{\mathfrak{s}} = z_0^a p_{\mathfrak{s}}$ and $z_0 \nmid p_{\mathfrak{s}}$. Let $w_k$ be the Whitney numbers of the first kind of the intersection lattice of $\mathcal{A}_{\mathfrak{s}}^*$. Then*

$$P_{\mathfrak{s}}(t) = \sum_{k \geq 0} w_k t^k.$$

*Equivalently, $b_k(\mathfrak{s}) = w_k$ for every $k \geq 0$.*

*Proof.* The product decomposition

$$(V_{\mathfrak{s}}^*)^c \cong \mathbb{C}^{\dim \mathfrak{n}} \times M(\mathcal{A}_{\mathfrak{s}}^*)$$

shows that $(V_{\mathfrak{s}}^*)^c$ and $M(\mathcal{A}_{\mathfrak{s}}^*)$ have the same cohomology. By the Orlik–Solomon theorem, this cohomology is the Orlik–Solomon algebra of $\mathcal{A}_{\mathfrak{s}}^*$. The Orlik–Solomon formula gives

$$P_{\mathfrak{s}}(t) = \sum_{X \in L(\mathcal{A}_{\mathfrak{s}}^*)} \mu(X)(-t)^{\operatorname{rank}(X)}.$$

For a geometric lattice, $\mu(X)$ has sign $(-1)^{\operatorname{rank}(X)}$. Therefore the coefficient of $t^k$ is $\sum_{\operatorname{rank}(X)=k} |\mu(X)| = w_k$. □

Thus the Betti numbers $b_j(\mathfrak{s})$ are determined by the spectral matroid, answering Problem 1.14.

We may now determine the degree of $P_{\mathfrak{s}}(t)$ from the nilradical weights. This refines Proposition 1.13 by identifying the rank which gives the degree.

**Theorem 2.21.** *Let $\Delta^\times$ be the set of distinct nonzero nilradical weights. Then*

$$\deg P_{\mathfrak{s}}(t) = \operatorname{rank}\{z_0 + \alpha : \alpha \in \Delta^\times\}.$$

*If $\Delta^\times \neq \varnothing$, then for any fixed $\alpha_0 \in \Delta^\times$, this rank is $1 + \dim \operatorname{span}\{\alpha - \alpha_0 : \alpha \in \Delta^\times\}$. If $\Delta^\times = \varnothing$, then $p_{\mathfrak{s}} = 1$ and $P_{\mathfrak{s}}(t) = 1$.*

*Proof.* By Theorem 2.20, the Poincaré polynomial of $(V_{\mathfrak{s}}^*)^c$ is the Poincaré polynomial of the Orlik–Solomon algebra of the matroid represented by

$$\Phi_{\mathfrak{s}}^* = \{z_0 + \alpha : \alpha \in \Delta^\times\}.$$

Hence the degree of $P_{\mathfrak{s}}(t)$ is the rank of this matroid, namely the dimension of the vector space spanned by the forms in $\Phi_{\mathfrak{s}}^*$. If $\Delta^\times = \varnothing$, then there are no non-$z_0$ spectral factors, so $p_{\mathfrak{s}} = 1$ and $P_{\mathfrak{s}}(t) = 1$.

Assume now that $\Delta^\times \neq \varnothing$ and fix $\alpha_0 \in \Delta^\times$. Then

$$z_0 + \alpha = (z_0 + \alpha_0) + (\alpha - \alpha_0)$$

for every $\alpha \in \Delta^\times$. Therefore

$$\operatorname{span}\{z_0 + \alpha : \alpha \in \Delta^\times\} = \mathbb{C}(z_0 + \alpha_0) + \operatorname{span}\{\alpha - \alpha_0 : \alpha \in \Delta^\times\}.$$

The second summand lies in $V_f^*$, while $z_0 + \alpha_0$ has nonzero $z_0$-component, so the sum is direct. Hence

$$\operatorname{rank}\{z_0 + \alpha : \alpha \in \Delta^\times\} = 1 + \dim \operatorname{span}\{\alpha - \alpha_0 : \alpha \in \Delta^\times\}.$$

This is one plus the dimension of the affine span of the nonzero weight configuration. The result follows. □

*Remark* 2.22. It is important that the degree depends on the affine span of the nonzero weights, not on their ordinary linear span. For example, if $\dim V_f = 1$ and $\Delta^\times = \{\alpha\}$ with $\alpha \neq 0$, then $\Delta^\times$ spans $V_f^*$ as a vector space, but

$$\operatorname{rank}\{z_0 + \alpha\} = 1,$$

so $P_{\mathfrak{s}}(t)$ has degree 1, not 2.

The preceding theorem gives a direct upper bound for the Betti numbers. Let $m^* = k(\mathfrak{s}) - 1$. This is the number of distinct linear factors of $p_{\mathfrak{s}}$, or equivalently the number of elements in the ground set $\Phi_{\mathfrak{s}}^*$ of the spectral matroid.

**Corollary 2.23.** *Let $k(\mathfrak{s})$ be the number of distinct linear factors in $Q_{\mathfrak{s}}(z_0, \mathbf{z})$. Then, for all $j \geq 0$,*

$$b_j(\mathfrak{s}) \leq \binom{k(\mathfrak{s}) - 1}{j}.$$

*Proof.* Let $m^* = k(\mathfrak{s}) - 1$. By Corollary 2.12, the cohomology of $(V_{\mathfrak{s}}^*)^c$ is isomorphic to a quotient $E/I$, where $E$ is the exterior algebra generated by the $m^*$ distinct factors in $\Phi_{\mathfrak{s}}^*$ and $I$ is the ideal generated by the relations coming from dependent subsets of $\Phi_{\mathfrak{s}}^*$. Therefore the degree-$j$ cohomology is a quotient of the degree-$j$ part of $E$. Since $\dim_{\mathbb{C}} E^j = \binom{m^*}{j}$, we obtain

$$b_j(\mathfrak{s}) \leq \binom{m^*}{j} = \binom{k(\mathfrak{s}) - 1}{j}.$$

□

We conclude this subsection by establishing a qualitative property of the Betti numbers of $(V_{\mathfrak{s}}^*)^c$.

Recall that a sequence of non-negative real numbers $a_0, a_1, \ldots, a_n$ is log-concave if $a_k^2 \geq a_{k-1}a_{k+1}$ for $1 \leq k < n$. If the sequence has no internal zeros, log-concavity implies unimodality. To prove log-concavity for the Betti numbers, we use the characteristic polynomial of the spectral matroid.

**Definition 2.24** (Characteristic Polynomial of a Matroid)**.** Let $\mathcal{M}$ be a matroid of rank $r$. The **characteristic polynomial** of $\mathcal{M}$ is defined via the Möbius function of its lattice as:

$$\chi_{\mathcal{M}}(q) = \sum_{X \in L(\mathcal{M})} \mu(\hat{0}, X) q^{r - \operatorname{rank}(X)}.$$

We use the following result from algebraic geometry concerning these polynomials.

**Lemma 2.25** ([HK12])**.** *Let $\mathcal{M}$ be a matroid representable over a field of characteristic 0. Then the absolute values of the coefficients of its characteristic polynomial form a log-concave sequence.*

We now connect our topological invariant to this algebraic object.

**Lemma 2.26.** *Let $\mathcal{M}_{\mathfrak{s}}^*$ be the matroid represented by the distinct linear factors of $p_{\mathfrak{s}}$. If $r = \operatorname{rank} \mathcal{M}_{\mathfrak{s}}^*$, then*

$$\chi_{\mathcal{M}_{\mathfrak{s}}^*}(q) = q^r P_{\mathfrak{s}}(-q^{-1}).$$

*Equivalently, the Betti numbers of $(V_{\mathfrak{s}}^*)^c$ are the absolute values of the coefficients of the ordinary characteristic polynomial $\chi_{\mathcal{M}_{\mathfrak{s}}^*}(q)$.*

*Proof.* Using the definition of the Whitney numbers

$$w_k = \sum_{\operatorname{rank}(X) = k} |\mu(\hat{0}, X)|,$$

and the sign property $\mu(\hat{0}, X) = (-1)^{\operatorname{rank}(X)} |\mu(\hat{0}, X)|$ for geometric lattices, the characteristic polynomial of $\mathcal{M}_{\mathfrak{s}}^*$ is

$$\chi_{\mathcal{M}_{\mathfrak{s}}^*}(q) = \sum_{k=0}^{r} (-1)^k w_k q^{r-k}.$$

Since

$$P_{\mathfrak{s}}(t) = \sum_{k=0}^{r} w_k t^k,$$

we obtain

$$q^r P_{\mathfrak{s}}(-q^{-1}) = q^r \sum_{k=0}^{r} w_k(-q^{-1})^k = \sum_{k=0}^{r}(-1)^k w_k q^{r-k} = \chi_{\mathcal{M}_{\mathfrak{s}}^*}(q).$$

Thus the Betti numbers are the absolute values of the coefficients of $\chi_{\mathcal{M}_{\mathfrak{s}}^*}(q)$. ☐

**Corollary 2.27.** *Let $r = \operatorname{rank} \mathcal{M}_{\mathfrak{s}}^*$. Then the Betti numbers $b_0(\mathfrak{s}), \ldots, b_r(\mathfrak{s})$ form a log-concave sequence:*

$$b_{j-1}(\mathfrak{s})\, b_{j+1}(\mathfrak{s}) \leq b_j(\mathfrak{s})^2 \qquad 1 \leq j \leq r-1.$$

*Moreover, $b_j(\mathfrak{s}) = 0$ for $j > r$.*

*Proof.* The spectral matroid $\mathcal{M}_{\mathfrak{s}}^*$ is represented by linear forms over $\mathbb{C}$, hence is representable over a field of characteristic 0. By Lemma 2.25, the absolute values of the coefficients of $\chi_{\mathcal{M}_{\mathfrak{s}}^*}(q)$ form a log-concave sequence. By Lemma 2.26, these absolute values are the Betti numbers $b_j(\mathfrak{s})$ for $0 \leq j \leq r$. The vanishing for $j > r$ follows from Theorem 2.21. ☐

## 3. Solvable Lie Algebras with Heisenberg Nilradical

3.1. **The Spectral Invariant $k(L)$.** We now specialize to solvable Lie algebras $\mathfrak{s}$ such that $\mathrm{Nil}(\mathfrak{s}) \cong \mathfrak{h}(m)$, where $\mathfrak{h}(m)$ has dimension $2m+1$. If $\mathfrak{s}$ has dimension $2m+n+1$, we say that $\mathfrak{s}$ has nilradical $\mathfrak{h}(m)$ and extension dimension $n$. The low-dimensional cases corresponding to $\mathfrak{h}(1)$ and $\mathfrak{h}(2)$ were classified by Rubin and Winternitz into discrete and parameterized families. The findings of [RW93] can be summarized as follows:

| $(\dim \mathfrak{h}, n)$ | Total Dimension | Number of Families | 1-Param Families | 2-Param Families |
|---|---|---|---|---|
| (3, 1) | 4 | 3 | 1 | 0 |
| (3, 2) | 5 | 1 | 0 | 0 |
| (5, 1) | 6 | 8 | 3 | 1 |
| (5, 2) | 7 | 8 | 2 | 1 |
| (5, 3) | 8 | 1 | 0 | 0 |

We denote the $t$-th family of solvable extensions with Heisenberg nilradical $\mathfrak{h}(m)$ and extension dimension $n$, depending on $s$ parameters, by $\mathfrak{s}_{2m+1,n}^{s,t}$.

Our study relies on the structure theorem of [RW93], which provides a canonical normal form for all solvable extensions of the Heisenberg algebra.

**Theorem 3.1** ([RW93])**.** *Every indecomposable solvable Lie algebra $\mathfrak{s}$ over $\mathbb{F} = \mathbb{C}$ or $\mathbb{F} = \mathbb{R}$ whose nilradical is the Heisenberg algebra $\mathfrak{h}(m)$ admits a canonical basis*

$$(h, p_1, \ldots, p_m, q_1, \ldots, q_m, f_1, \ldots, f_k),$$

*with commutation relations given, in addition to those of $\mathfrak{h}(m)$, by*

$$([f_\alpha, h], [f_\alpha, \xi]) = (h, \xi) M_\alpha, \qquad [f_\alpha, f_\beta] = r_{\alpha\beta} h,$$

*where $\xi = (p_1, \ldots, p_m, q_1, \ldots, q_m)$ and*

$$M_\alpha = \begin{pmatrix} 2a_\alpha & 0 \\ 0 & a_\alpha I_{2m} + X_\alpha \end{pmatrix}.$$

*The constants satisfy $a_1 \in \{0, 1\}$ and $a_2 = \cdots = a_k = 0$, while the matrices $X_1, \ldots, X_k \in \mathfrak{sp}(2m, \mathbb{F})$ commute pairwise, $[X_\alpha, X_\beta] = 0$. If $a_1 = 0$ (resp. $a_1 = 1$), then $\{X_1, \ldots, X_k\}$ (resp. $\{X_2, \ldots, X_k\}$) is linearly nilindependent. Furthermore, the constants $r_{\alpha\beta}$ satisfy $r_{\alpha\beta} = 0$ for all $\alpha, \beta$ when $a_1 = 1$, and $r_{\alpha\beta} = -r_{\beta\alpha} \in \mathbb{F}$ for all $\alpha, \beta$ when $a_1 = 0$.*

This normal form allows us to separate the contribution of the extension from the nontrivial action on the nilradical.

**Lemma 3.2.** *Let $\mathfrak{s}$ be a solvable Lie algebra with Heisenberg nilradical $\mathfrak{h}(m)$ and $k$-dimensional extension $V_f$. Then the characteristic polynomial factors as*

$$Q_{\mathfrak{s}}(z) = z_0^k \left( z_0 + \sum_{\alpha=1}^{k} 2a_\alpha z_{f_\alpha} \right) \det\left( z_0 I_{2m} + \sum_{\alpha=1}^{k} z_{f_\alpha}(a_\alpha I_{2m} + X_\alpha) \right).$$

*Proof.* Let $\mathfrak{s} = \mathfrak{n} \oplus V_f$, where $\mathfrak{n} = \mathfrak{h}(m)$ and $V_f = \operatorname{span}\{f_1, \dots, f_k\}$. We apply Theorem 2.4. Since $\dim V_f = k$, the quotient contribution is $z_0^k$.

It remains only to identify the weights of the action of $V_f$ on the nilradical. By Theorem 3.1, for each $\alpha$, the restriction of $\operatorname{ad}(f_\alpha)$ to $\mathfrak{n}$ has the form $\operatorname{diag}(2a_\alpha, a_\alpha I_{2m} + X_\alpha)$, with respect to the decomposition $\mathfrak{n} = \operatorname{span}\{h\} \oplus \operatorname{span}\{p_i, q_i\}_{i=1}^m$. Therefore, on $\operatorname{span}\{h\}$ the corresponding weight is $\ell_h(z_f) = 2\sum_{\alpha=1}^k a_\alpha z_{f_\alpha}$. On the complementary $2m$-dimensional subspace, the contribution is the determinant of $z_0 I_{2m} + \sum_{\alpha=1}^k z_{f_\alpha}(a_\alpha I_{2m} + X_\alpha)$.

Thus Theorem 2.4 gives

$$Q_{\mathfrak{s}}(z) = z_0^k \left( z_0 + 2\sum_{\alpha=1}^{k} a_\alpha z_{f_\alpha} \right) \det\left( z_0 I_{2m} + \sum_{\alpha=1}^{k} z_{f_\alpha}(a_\alpha I_{2m} + X_\alpha) \right),$$

as desired. ☐

Using the symmetry of the Heisenberg nilradical, we obtain a factorization of the characteristic polynomial:

**Theorem 3.3.** *Let $\mathfrak{s}$ be a solvable Lie algebra with Heisenberg nilradical $\mathfrak{n} \cong \mathfrak{h}(m)$, and let $D(z) = \sum_{\alpha=1}^k z_\alpha \operatorname{ad}(f_\alpha)|_{\mathfrak{n}}$. Set $\gamma(z) = \frac{1}{m+1} \operatorname{Tr} D(z)$. Then*

$$Q_{\mathfrak{s}}(z_0, z) = z_0^k (z_0 + \gamma(z)) \prod_{i=1}^{m} \left( z_0 + \frac{\gamma(z)}{2} - \delta_i(z) \right) \left( z_0 + \frac{\gamma(z)}{2} + \delta_i(z) \right),$$

*where $\delta_1, \dots, \delta_m \in V_f^*$ are linear forms such that the weights of the commuting family $X_1, \dots, X_k$ on $\mathfrak{n}/Z(\mathfrak{n})$ are $\{\pm\delta_1, \dots, \pm\delta_m\}$, counted with multiplicity.*

*Proof.* By Lemma 3.2, we have

$$Q_{\mathfrak{s}}(z_0, z) = z_0^k \left( z_0 + 2\sum_{\alpha=1}^{k} a_\alpha z_\alpha \right) \det\left( z_0 I_{2m} + \sum_{\alpha=1}^{k} z_\alpha (a_\alpha I_{2m} + X_\alpha) \right),$$

where $X_\alpha \in \mathfrak{sp}(2m, \mathbb{C})$. Put $\gamma(z) = 2\sum_{\alpha=1}^k a_\alpha z_\alpha$ and $S(z) = \sum_{\alpha=1}^k z_\alpha X_\alpha$. Then the determinant factor becomes $\det\left( z_0 I_{2m} + \frac{\gamma(z)}{2} I_{2m} + S(z) \right)$.

Since the matrices $X_1, \dots, X_k$ commute, they are simultaneously triangularizable over $\mathbb{C}$. In a common triangular basis, the diagonal entries of $S(z) = \sum_{\alpha=1}^k z_\alpha X_\alpha$ are linear forms in $z_1, \dots, z_k$. The symplectic form identifies the standard $\mathfrak{sp}(2m, \mathbb{C})$-module with its dual. Passing to the dual changes each weight to its negative, while self-duality preserves the weight multiset. Therefore the weights for the commuting family $X_1, \dots, X_k$ occur in opposite pairs. After reordering, the diagonal entries of $S(z)$ may be written as $\{\pm\delta_1(z), \dots, \pm\delta_m(z)\}$, where each $\delta_i$ is a linear form. Therefore

$$\det\left( z_0 I_{2m} + \frac{\gamma(z)}{2} I_{2m} + S(z) \right) = \prod_{i=1}^{m} \left( z_0 + \frac{\gamma(z)}{2} - \delta_i(z) \right) \left( z_0 + \frac{\gamma(z)}{2} + \delta_i(z) \right).$$

It remains to prove the expression of $\gamma(z)$ as a trace. The action of $D(z)$ on $Z(\mathfrak{n})$ has trace $\gamma(z)$, while its induced action on $\mathfrak{n}/Z(\mathfrak{n})$ is $\frac{\gamma(z)}{2} I_{2m} + S(z)$. Since $S(z)$ is traceless, the quotient contributes $m\gamma(z)$ to the trace. Therefore $\operatorname{Tr} D(z) = (m+1)\gamma(z)$, so $\gamma(z) = \frac{1}{m+1} \operatorname{Tr} D(z)$. This proves the result. ☐

This gives us a simpler formula for $k(\mathfrak{s})$:

**Corollary 3.4.** *For a solvable extension of $\mathfrak{h}(m)$, the spectral invariant $k(\mathfrak{s})$ is given by*

$$k(\mathfrak{s}) = \left| \{0, \gamma\} \cup \bigcup_{i=1}^{m} \left\{ \frac{\gamma}{2} - \delta_i, \frac{\gamma}{2} + \delta_i \right\} \right|,$$

*where the linear forms are defined as in Theorem 3.3. In particular, we have the sharp bound $k(\mathfrak{s}) \leq 2m + 2$.*

Now, we give a criterion that characterizes when a family of solvable Heisenberg extensions is spectrally equivalent.

**Theorem 3.5.** *Let $\mathfrak{s}$ and $\mathfrak{s}'$ be same-dimensional solvable Lie algebras with Heisenberg nilradical $\mathfrak{h}(m)$ and complements $V_f$ and $V_f'$. Let $\gamma, \gamma'$ and $\{\delta_i\}_{i=1}^m, \{\delta_j'\}_{j=1}^m$ be the linear forms appearing in Theorem 3.3 for $\mathfrak{s}$ and $\mathfrak{s}'$, respectively. Then $\mathfrak{s}$ and $\mathfrak{s}'$ are spectrally equivalent if and only if their total $z_0$-multiplicities agree and there exists a linear isomorphism between the spans of the corresponding nonzero Heisenberg weight multisets carrying $\{\gamma\}^\times \cup \{\gamma/2 - \delta_i, \gamma/2 + \delta_i : 1 \leq i \leq m\}^\times$ onto $\{\gamma'\}^\times \cup \{\gamma'/2 - \delta_j', \gamma'/2 + \delta_j' : 1 \leq j \leq m\}^\times$ as multisets, where the superscript $^\times$ means that zero forms are omitted.*

*Proof.* By Theorem 3.3, the nilradical weight multiset of $\mathfrak{s}$ is $\{\gamma\} \cup \{\gamma/2 - \delta_i, \gamma/2 + \delta_i : 1 \leq i \leq m\}$, counted with multiplicity. Zero forms in this multiset contribute to the total exponent of the factor $z_0$, while the remaining forms give the nonzero nilradical weight multiset $\Delta_{\mathfrak{s}}^\times$. The same description holds for $\mathfrak{s}'$.

Therefore the stated criterion is Proposition 2.7 applied to these Heisenberg weight multisets. □

We can also specialize Corollary 2.10 to the Heisenberg nilradical setting.

**Theorem 3.6.** *Let $\mathfrak{s}(p)$ be a family of solvable Lie algebras with Heisenberg nilradical $\mathfrak{h}(m)$. Assume that the total multiplicity of the factor $z_0$ is constant throughout the family, and that the zero entries in the Heisenberg weight multiset*

$$\{\gamma\} \cup \{\gamma/2 - \delta_j(p), \gamma/2 + \delta_j(p) : 1 \leq j \leq m\}$$

*occur in the same labeled positions as $p$ varies. Suppose that, for some $i$, one has $\delta_i(p) = \mu(p)\gamma + \tau$, where $\gamma$, $\tau$, and the remaining $\delta_j$ with $j \neq i$ are fixed as $p$ varies, and where $\tau \notin \operatorname{span}\{\gamma, \delta_j : j \neq i\}$. Then changing $\mu(p)$ gives a spectrally equivalent member of the family. If such a change gives a non-isomorphic Lie algebra, then the family is not spectrally rigid.*

*Proof.* Let $p$ and $p'$ be two parameter values which differ only in the value of $\mu$, and put $\Delta\mu = \mu(p') - \mu(p)$. By Theorem 3.3 and Proposition 2.7, spectral equivalence is determined by the total $z_0$-multiplicity and by the nonzero Heisenberg weight multiset.

Let $W = \operatorname{span}\{\gamma, \delta_j : j \neq i\}$. By hypothesis, $\tau \notin W$. Choose a basis of $W$, extend it by $\tau$, and then extend further to a basis of $V_f^*$. Define $T \in \mathrm{GL}(V_f^*)$ by fixing $W$ pointwise, sending $T(\tau) = \tau + \Delta\mu\,\gamma$, and fixing all remaining basis elements. Then $T(\delta_i(p)) = T(\mu(p)\gamma + \tau) = \mu(p')\gamma + \tau = \delta_i(p')$.

Since $T$ fixes $\gamma$ and every $\delta_j$ with $j \neq i$, it maps each labeled form $\gamma/2 \pm \delta_j(p)$ to the corresponding labeled form $\gamma/2 \pm \delta_j(p')$. By the hypothesis on zero entries, omitting the zero forms before and after applying $T$ gives the same nonzero multiset. Hence $T$ carries the nonzero Heisenberg weight multiset for $p$ onto the corresponding nonzero multiset for $p'$. Proposition 2.7 gives $\mathfrak{s}(p) \sim \mathfrak{s}(p')$.

Thus varying $\mu$ does not change the spectral equivalence class. If the resulting Lie algebras are not isomorphic, the family contains non-isomorphic spectrally equivalent members, and is therefore not spectrally rigid. □

3.2. **Topological Spectral Invariants.** We now specialize the topological results of Section 2.2 to solvable Lie algebras with Heisenberg nilradical. For Heisenberg nilradicals, Theorem 3.3 shows that the nilradical weights are $\gamma$ and $\gamma/2 \pm \delta_i$ for $1 \leq i \leq m$, with repetitions and possible coincidences allowed. The factor $z_0$ is treated separately: it is removed when passing from $Q_{\mathfrak{s}}$ to $p_{\mathfrak{s}}$. Thus the topology is controlled by the span of the forms $\gamma, \delta_1, \ldots, \delta_m$, and the paired factors can force circuits in the spectral matroid when the corresponding paired forms are distinct and the relevant $\delta_i$ are linearly independent. These two features refine the degree bound and the binomial bounds from the general theory.

Throughout this subsection, $\Phi^*$ denotes the ground set of the matroid associated to $p_{\mathfrak{s}}$. Thus $\Phi^*$ consists of the distinct forms $z_0+\lambda$, where $\lambda$ ranges over the distinct nonzero nilradical weights, and $|\Phi^*| = k(\mathfrak{s})-1$. When we refer to a flat $F$ of this matroid and write $|F|$, we view $F$ as a subset of the ground set $\Phi^*$.

**Theorem 3.7.** *Let $\mathfrak{s}$ be a solvable Lie algebra with Heisenberg nilradical $\mathfrak{h}(m)$, and let $\gamma, \delta_1, \ldots, \delta_m$ be the linear forms appearing in Theorem 3.3. Let $\Phi^* = \{z_0 + \lambda : \lambda \in \Lambda^\times\}$, where $\Lambda^\times$ is the set of distinct nonzero Heisenberg nilradical weights. Let $r^* = \operatorname{rank}\Phi^*$. Then*

$$P_{\mathfrak{s}}(t) = \sum_{i=0}^{r^*} w_i t^i.$$

*Consequently,* $\deg P_{\mathfrak{s}}(t) = r^*$*, so* $b_j(\mathfrak{s}) = 0$ *for* $j > r^*$*. Moreover,*

$$r^* \le 1 + \dim \operatorname{span}\{\gamma, \delta_1, \ldots, \delta_m\} \le m + 2,$$

*and therefore* $b_j(\mathfrak{s}) = 0$ *for* $j > m + 2$*.*

*Proof.* The formula $P_{\mathfrak{s}}(t) = \sum_{i=0}^{r^*} w_i t^i$ is Theorem 2.20 applied to the arrangement represented by $\Phi^*$. Hence $\deg P_{\mathfrak{s}}(t) = r^*$ and $b_j(\mathfrak{s}) = 0$ for $j > r^*$.

It remains to bound $r^*$. If $\Lambda^\times = \varnothing$, then $\Phi^* = \varnothing$ and $r^* = 0$. Otherwise, choose $\lambda_0 \in \Lambda^\times$. By Theorem 2.21,

$$r^* = 1 + \dim \operatorname{span}\{\lambda - \lambda_0 : \lambda \in \Lambda^\times\}.$$

Every nonzero Heisenberg nilradical weight lies among $\gamma$ and $\gamma/2 \pm \delta_i$ for $1 \le i \le m$. Therefore every difference $\lambda - \lambda_0$ lies in $\operatorname{span}\{\gamma, \delta_1, \ldots, \delta_m\}$. Thus

$$r^* \le 1 + \dim \operatorname{span}\{\gamma, \delta_1, \ldots, \delta_m\} \le m + 2.$$

Therefore $b_j(\mathfrak{s}) = 0$ for $j > m + 2$. □

The next result gives a refinement of the general bound for $b_2$. It is useful because coincidences among rank-two flats occur in the Heisenberg cases and are not detected by $k(\mathfrak{s})$ alone.

**Proposition 3.8.** *Let $m^* = k(\mathfrak{s}) - 1 = |\Phi^*|$, and let $\mathcal{F}_2$ be the set of rank-two flats of the matroid represented by $\Phi^*$. For $F \in \mathcal{F}_2$, let $|F|$ be the number of elements of $\Phi^*$ contained in $F$. Then $b_1(\mathfrak{s}) = m^*$ and*

$$b_2(\mathfrak{s}) = \binom{m^*}{2} - \sum_{F \in \mathcal{F}_2} \binom{|F| - 1}{2}.$$

*Equivalently,* $b_1(\mathfrak{s}) = k(\mathfrak{s}) - 1$ *and*

$$b_2(\mathfrak{s}) = \binom{k(\mathfrak{s}) - 1}{2} - \sum_{F \in \mathcal{F}_2} \binom{|F| - 1}{2}.$$

*In particular,* $b_2(\mathfrak{s}) \le \binom{k(\mathfrak{s})-1}{2}$*, with equality if and only if every rank-two flat contains exactly two elements of* $\Phi^*$*.*

*Proof.* By Theorem 2.20, the Betti numbers are the Whitney numbers of the first kind of the matroid represented by $\Phi^*$. Since $|\Phi^*| = m^*$, we have $w_0 = 1$ and $w_1 = m^*$, so $b_1(\mathfrak{s}) = m^*$.

For a rank-two flat $F$, the interval $[\hat{0}, F]$ has $|F|$ atoms. Hence $\mu(F) = |F| - 1$ in absolute value, so $w_2 = \sum_{F \in \mathcal{F}_2} (|F| - 1)$. Every unordered pair of distinct elements of $\Phi^*$ spans a unique rank-two flat, and a flat $F$ containing $|F|$ elements accounts for $\binom{|F|}{2}$ such pairs. Therefore $\binom{m^*}{2} = \sum_{F \in \mathcal{F}_2} \binom{|F|}{2}$.

Using $|F| - 1 = \binom{|F|}{2} - \binom{|F|-1}{2}$, we get

$$b_2(\mathfrak{s}) = w_2 = \sum_{F \in \mathcal{F}_2} (|F| - 1) = \binom{m^*}{2} - \sum_{F \in \mathcal{F}_2} \binom{|F| - 1}{2}.$$

The final inequality and equality condition follow immediately. □

The same method gives a degree-three formula in terms of the rank-three flats. Further simplification would necessarily lose information about the incidence structure of the spectral matroid.

**Proposition 3.9.** *Let $\mathcal{F}_3$ be the set of rank-three flats of $\mathcal{M}^*_{\mathfrak{s}}$. If $X \in \mathcal{F}_3$, let $\mathcal{F}_2(X) = \{F \in \mathcal{F}_2 : F \subseteq X\}$. Then*

$$b_3(\mathfrak{s}) = \sum_{X \in \mathcal{F}_3} \left( 1 - |X| + \sum_{F \in \mathcal{F}_2(X)} (|F| - 1) \right).$$

*Proof.* By Theorem 2.20, $b_3 = w_3$. For a rank-three flat $X$, the Möbius relation gives

$$\mu(X) = -\left( 1 + \sum_{a \leq X} \mu(a) + \sum_{F \in \mathcal{F}_2(X)} \mu(F) \right).$$

There are $|X|$ atoms below $X$, each with Möbius value $-1$, and each rank-two flat $F \subseteq X$ has $\mu(F) = |F| - 1$. Therefore $|\mu(X)| = 1 - |X| + \sum_{F \in \mathcal{F}_2(X)} (|F| - 1)$, and hence

$$b_3(\mathfrak{s}) = \sum_{X \in \mathcal{F}_3} \left( 1 - |X| + \sum_{F \in \mathcal{F}_2(X)} (|F| - 1) \right).$$

□

The preceding formula is often the best way to compute $b_3$. For a closed-form improvement over Corollary 2.23, we use the circuits forced by the Heisenberg pairing.

**Lemma 3.10.** *If $i \neq j$ and $\delta_i, \delta_j$ are linearly independent, then the set $\{z_0 + \gamma/2 - \delta_i,\ z_0 + \gamma/2 + \delta_i,\ z_0 + \gamma/2 - \delta_j,\ z_0 + \gamma/2 + \delta_j\}$ is a circuit of rank 3, provided all four displayed forms are distinct and occur in $\Phi^*$.*

*Proof.* The relation $(z_0 + \frac{\gamma}{2} - \delta_i) + (z_0 + \frac{\gamma}{2} + \delta_i) - (z_0 + \frac{\gamma}{2} - \delta_j) - (z_0 + \frac{\gamma}{2} + \delta_j) = 0$ proves dependence. Since $\delta_i$ and $\delta_j$ are linearly independent, the forms $z_0 + \frac{\gamma}{2}$, $\delta_i$, and $\delta_j$ are linearly independent. Comparing coefficients in this basis shows that no three of the four displayed forms are dependent. Thus the displayed set is a circuit of rank 3. □

*Remark* 3.11. If $\gamma$ and $\delta_i$ are linearly independent and the three forms $z_0 + \gamma$, $z_0 + \gamma/2 - \delta_i$, and $z_0 + \gamma/2 + \delta_i$ occur in $\Phi^*$, then these three forms are independent. Thus removing the factor $z_0$ eliminates the corresponding dependence involving the form $z_0$.

Let $q_{\mathfrak{s}}$ be the dimension of the subspace of the degree-three Orlik–Solomon ideal spanned by the elements $\partial(e_C)$, where $C$ ranges over the circuits in Lemma 3.10. The following bound improves the general estimate whenever $q_{\mathfrak{s}} > 0$.

**Theorem 3.12.** *Let $m^* = |\Phi^*| = k(\mathfrak{s}) - 1$. Then*

$$b_3(\mathfrak{s}) \leq \binom{m^*}{3} - q_{\mathfrak{s}}.$$

*Equivalently,*

$$b_3(\mathfrak{s}) \leq \binom{k(\mathfrak{s}) - 1}{3} - q_{\mathfrak{s}}.$$

*Proof.* The degree-three part of the exterior algebra on the $m^*$ generators indexed by $\Phi^*$ has dimension $\binom{m^*}{3}$. Each circuit $C$ from Lemma 3.10 gives an Orlik–Solomon relation $\partial(e_C)$ in degree 3. By definition, these relations span a subspace of dimension $q_{\mathfrak{s}}$.

The degree-three cohomology is obtained from the degree-three exterior algebra by quotienting by the degree-three part of the Orlik–Solomon ideal. Therefore

$$b_3(\mathfrak{s}) \leq \binom{m^*}{3} - q_{\mathfrak{s}}.$$

Since $m^* = k(\mathfrak{s}) - 1$, this gives the equivalent form. □

The rank $q_{\mathfrak{s}}$ can be estimated directly from independent subsets of the forms $\delta_i$.

**Corollary 3.13.** *Suppose that $\delta_1, \dots, \delta_s$ are linearly independent and that the corresponding paired forms are distinct elements of $\Phi^*$. Then the circuits of Lemma 3.10 give $q_{\mathfrak{s}} \geq \binom{s}{2}$.*

*Proof.* For each pair $1 \leq i < j \leq s$, Lemma 3.10 gives a four-element circuit $C_{ij}$. Write its Orlik–Solomon boundary as $\partial(e_{C_{ij}})$. To prove that these boundary elements are linearly independent, suppose that $\sum_{1 \leq i < j \leq s} a_{ij} \partial(e_{C_{ij}}) = 0$.

Fix a pair $\{i, j\}$. The boundary $\partial(e_{C_{ij}})$ contains a degree-three term involving both elements from the $i$-pair and one element from the $j$-pair. No boundary $\partial(e_{C_{uv}})$ with $\{u, v\} \neq \{i, j\}$ contains this same term, because such a boundary only involves the elements indexed by $u$ and $v$. Therefore the coefficient of this term in the above linear combination is $a_{ij}$, up to sign. Hence $a_{ij} = 0$. Since the pair $\{i, j\}$ was arbitrary, all coefficients $a_{ij}$ vanish.

Thus the elements $\partial(e_{C_{ij}})$ are linearly independent. Hence the circuits contribute at least $\binom{s}{2}$ independent degree-three Orlik–Solomon relations, so $q_{\mathfrak{s}} \geq \binom{s}{2}$. ☐

Finally, the log-concavity theorem improves the higher-degree bounds once the degree-three correction is known. Let $B_3 = \binom{k(\mathfrak{s})-1}{3} - q_{\mathfrak{s}}$ and $B_2 = b_2(\mathfrak{s})$.

**Corollary 3.14.** *Assume $B_2 > 0$. For every $j \geq 3$,*

$$b_j(\mathfrak{s}) \leq B_2 \left( \min \left\{ \frac{B_2}{b_1(\mathfrak{s})}, \frac{B_3}{B_2} \right\} \right)^{j-2}.$$

*Together with Theorem 3.7, one also has $b_j(\mathfrak{s}) = 0$ for $j > r^*$, where $r^* = \operatorname{rank} \Phi^*$.*

*Proof.* Theorem 3.12 gives $b_3 \leq B_3$, and Proposition 3.8 gives $B_2 = b_2$. By Corollary 2.27, the Betti numbers form a log-concave sequence. Since the Betti numbers of a matroid arrangement have no internal zeros, the ratios $b_j/b_{j-1}$ are nonincreasing for $1 \leq j \leq r^*$, where $r^* = \operatorname{rank} \Phi^*$. Therefore, for $3 \leq j \leq r^*$, one has $b_j/b_{j-1} \leq b_3/b_2 \leq B_3/B_2$ and also $b_j/b_{j-1} \leq b_2/b_1$. Iterating from degree 2 gives the displayed inequality. For $j > r^*$, Theorem 3.7 gives $b_j(\mathfrak{s}) = 0$. ☐

### 3.3. A Worked Example: The Algebra $\mathfrak{s}_{5,3}^{0,1}$.

We now give a worked example illustrating how the characteristic polynomial, spectral invariant, spectral matroid, and Poincaré polynomial are computed from the extension data in the classification of Rubin and Winternitz [RW93]. Consider the Lie algebra $\mathfrak{s} = \mathfrak{s}_{5,3}^{0,1}$. Here the nilradical is $\mathfrak{n} = \mathfrak{h}(2) = \operatorname{span}\{h, p_1, p_2, q_1, q_2\}$, with nonzero Heisenberg brackets $[p_1, q_1] = h$ and $[p_2, q_2] = h$. The extension space is $V_f = \operatorname{span}\{f_1, f_2, f_3\}$, so $\mathfrak{s} = \mathfrak{n} \oplus V_f$. We use the ordered basis $\mathcal{B} = (h, p_1, p_2, q_1, q_2, f_1, f_2, f_3)$. Thus the variables corresponding to $f_1, f_2, f_3$ are $z_6, z_7, z_8$, respectively.

For this algebra, the extension data are $a_1 = 1$ and $a_2 = a_3 = 0$, with $X_1 = 0$, $X_2 = \operatorname{diag}(1, 0, -1, 0)$, and $X_3 = \operatorname{diag}(0, 1, 0, -1)$, where the matrices $X_i$ act on $\operatorname{span}\{p_1, p_2, q_1, q_2\}$. Thus $D_i := \operatorname{ad}(f_i)|_{\mathfrak{n}}$ has the following form with respect to the ordered basis $(h, p_1, p_2, q_1, q_2)$:

$$D_1 = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \qquad D_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

and

$$D_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}.$$

Equivalently, the nonzero brackets involving the extension are $[f_1, h] = 2h$, $[f_1, p_i] = p_i$, and $[f_1, q_i] = q_i$ for $i = 1, 2$, together with $[f_2, p_1] = p_1$, $[f_2, q_1] = -q_1$, $[f_3, p_2] = p_2$, and $[f_3, q_2] = -q_2$. The extension is abelian in this case, so $[f_i, f_j] = 0$ for all $i, j$.

The combined action on the nilradical is $D(z_6, z_7, z_8) = z_6 D_1 + z_7 D_2 + z_8 D_3$. Using the matrices above, we get

$$D(z_6, z_7, z_8) = \begin{pmatrix} 2z_6 & 0 & 0 & 0 & 0 \\ 0 & z_6 + z_7 & 0 & 0 & 0 \\ 0 & 0 & z_6 + z_8 & 0 & 0 \\ 0 & 0 & 0 & z_6 - z_7 & 0 \\ 0 & 0 & 0 & 0 & z_6 - z_8 \end{pmatrix}.$$

Therefore the nilradical weights are $2z_6$, $z_6 + z_7$, $z_6 - z_7$, $z_6 + z_8$, and $z_6 - z_8$. Since $\dim V_f = 3$, the quotient $\mathfrak{s}/\mathfrak{n} \cong V_f$ contributes the factor $z_0^3$. Hence

$$\begin{aligned} Q_{\mathfrak{s}}(z_0, z_6, z_7, z_8) &= z_0^3 \det\big(z_0 I_5 + D(z_6, z_7, z_8)\big) \\ &= z_0^3 (z_0 + 2z_6)(z_0 + z_6 + z_7)(z_0 + z_6 - z_7)(z_0 + z_6 + z_8)(z_0 + z_6 - z_8). \end{aligned}$$

Thus the distinct linear factors are $z_0$, $z_0 + 2z_6$, $z_0 + z_6 - z_7$, $z_0 + z_6 + z_7$, $z_0 + z_6 - z_8$, and $z_0 + z_6 + z_8$. Therefore $k(\mathfrak{s}) = 6$. We now compute the Poincaré polynomial: Since

$$Q_{\mathfrak{s}}(z_0, z_6, z_7, z_8) = z_0^3 (z_0 + 2z_6)(z_0 + z_6 + z_7)(z_0 + z_6 - z_7)(z_0 + z_6 + z_8)(z_0 + z_6 - z_8),$$

we remove the factor $z_0^3$ and work with the central arrangement defined by the distinct linear factors of

$$p_{\mathfrak{s}}(z_0, z_6, z_7, z_8) = (z_0 + 2z_6)(z_0 + z_6 + z_7)(z_0 + z_6 - z_7)(z_0 + z_6 + z_8)(z_0 + z_6 - z_8).$$

Set

$$\begin{aligned} &L_1 = z_0 + 2z_6, \qquad L_2 = z_0 + z_6 - z_7, \quad L_3 = z_0 + z_6 + z_7, \\ &L_4 = z_0 + z_6 - z_8, \quad L_5 = z_0 + z_6 + z_8. \end{aligned}$$

The spectral matroid is represented by the coefficient vectors of these five linear forms in the vector space with coordinates $(z_0, z_6, z_7, z_8)$:

$$L_1 = (1, 2, 0, 0), \qquad L_2 = (1, 1, -1, 0), \qquad L_3 = (1, 1, 1, 0),$$
$$L_4 = (1, 1, 0, -1), \qquad L_5 = (1, 1, 0, 1).$$

These five vectors span a four-dimensional vector space, so the spectral matroid has rank 4.

We compute the Whitney numbers $w_i$ of the first kind. First, $w_0 = 1$. There are five rank-one flats, one for each linear form, so $w_1 = 5$. No two coefficient vectors are proportional, and no three coefficient vectors span a two-dimensional subspace. Hence every pair gives a rank-two flat, so $w_2 = \binom{5}{2} = 10$.

The rank-three flats are the first place where nontrivial incidence appears. There are six rank-three flats containing exactly three elements:

$$\{L_1, L_2, L_3\}, \quad \{L_1, L_2, L_4\}, \quad \{L_1, L_2, L_5\},$$
$$\{L_1, L_3, L_4\}, \quad \{L_1, L_3, L_5\}, \quad \{L_1, L_4, L_5\}.$$

There is also one rank-three flat containing four elements, namely $\{L_2, L_3, L_4, L_5\}$. For a rank-three flat $F$ containing $s$ atoms, the Möbius value is $\mu(F) = -(1 - s + \binom{s}{2})$. Thus a rank-three flat containing three atoms has Möbius value $-1$, while the rank-three flat containing four atoms has Möbius value $-3$. Hence $w_3 = 6 \cdot 1 + 1 \cdot 3 = 9$.

Finally, the top flat has rank 4. Its Möbius value is determined by $\sum_{X \leq \hat{1}} \mu(X) = 0$. Therefore

$$\mu(\hat{1}) = -\left(1 + 5(-1) + 10(1) + 6(-1) + 1(-3)\right) = 3.$$

So $w_4 = 3$. The Whitney numbers are therefore

$$w_0 = 1, \qquad w_1 = 5, \qquad w_2 = 10, \qquad w_3 = 9, \qquad w_4 = 3.$$

Therefore the Poincaré polynomial is

$$P_{\mathfrak{s}}(t) = \sum_{i=0}^{4} w_i t^i = 1 + 5t + 10t^2 + 9t^3 + 3t^4.$$

Equivalently, the Betti numbers are

$$b_0 = 1, \qquad b_1 = 5, \qquad b_2 = 10, \qquad b_3 = 9, \qquad b_4 = 3.$$

3.4. **Computations.** Using the Rubin–Winternitz list [RW93], we compute $Q_{\mathfrak{s}}(z)$, $k(\mathfrak{s})$, and $P_{\mathfrak{s}}(t)$ for the solvable extensions of $\mathfrak{h}(1)$ and $\mathfrak{h}(2)$ appearing below. The parameter restrictions are those of the Rubin–Winternitz normal forms. In the parameter rows, the listed values of $k(\mathfrak{s})$ and $P_{\mathfrak{s}}(t)$ apply under the restrictions displayed in the parameter column. Values at which one of the displayed weights becomes zero, or at which two displayed factors coincide, are listed separately.

For the family $\mathfrak{s}_{5,2}^{2,1,b,c}$, we use the notation

$$\mathcal{E}_b = \{0,\ b+1,\ b-1,\ 1-b,\ -1-b\}.$$

These are the values of $c$ for which the five non-$z_0$ factors

$$z_0 + 2z_6, \quad z_0 + z_6 - z_7, \quad z_0 + z_6 + z_7, \quad z_0 + (1-b)z_6 - cz_7, \quad z_0 + (1+b)z_6 + cz_7$$

contain a three-element dependent subset. Thus these values have a different spectral matroid from the regular values $c \notin \mathcal{E}_b$.

Table 1: Characteristic polynomials and spectral invariants for low-dimensional solvable Lie algebras with Heisenberg nilradical

| **Dim** | **Algebra** $\mathfrak{s}$ | $Q_{\mathfrak{s}}(z)$ | $k(\mathfrak{s})$ | $P_{\mathfrak{s}}(t)$ |
|---|---|---|---|---|
| 4 | $\mathfrak{s}_{3,1}^{0,1}$ | $z_0^2(z_0 - z_4)(z_0 + z_4)$ | 3 | $1 + 2t + t^2$ |
| 4 | $\mathfrak{s}_{3,1}^{0,2}$ | $z_0(z_0 + z_4)^2(z_0 + 2z_4)$ | 3 | $1 + 2t + t^2$ |
| 4 | $\mathfrak{s}_{3,1}^{1,1,b}$ | | | |
| | $b > 0,\ b \neq 1$ | $z_0(z_0 + 2z_4)(z_0 + (1-b)z_4)(z_0 + (1+b)z_4)$ | 4 | $1 + 3t + 2t^2$ |
| | $b = 0$ | $z_0(z_0 + 2z_4)(z_0 + z_4)^2$ | 3 | $1 + 2t + t^2$ |
| | $b = 1$ | $z_0^2(z_0 + 2z_4)^2$ | 2 | $1 + t$ |
| 5 | $\mathfrak{s}_{3,2}^{0,1}$ | $z_0^2(z_0 + 2z_4)(z_0 + z_4 - z_5)(z_0 + z_4 + z_5)$ | 4 | $1 + 3t + 3t^2 + t^3$ |
| 6 | $\mathfrak{s}_{5,1}^{0,1}$ | $z_0(z_0 + z_6)^4(z_0 + 2z_6)$ | 3 | $1 + 2t + t^2$ |
| 6 | $\mathfrak{s}_{5,1}^{0,2}$ | $z_0^4(z_0 - z_6)(z_0 + z_6)$ | 3 | $1 + 2t + t^2$ |
| 6 | $\mathfrak{s}_{5,1}^{0,3}$ | $z_0^2(z_0 - z_6)^2(z_0 + z_6)^2$ | 3 | $1 + 2t + t^2$ |
| 6 | $\mathfrak{s}_{5,1}^{0,4}$ | $z_0(z_0 + z_6)^4(z_0 + 2z_6)$ | 3 | $1 + 2t + t^2$ |
| 6 | $\mathfrak{s}_{5,1}^{1,1,c}$ | | | |
| | $c = 0$ | $z_0^4(z_0 - z_6)(z_0 + z_6)$ | 3 | $1 + 2t + t^2$ |
| | $0 < \lvert c\rvert \leq 1,\ c^2 \neq 1$ | $z_0^2(z_0 - z_6)(z_0 + z_6)(z_0 - cz_6)(z_0 + cz_6)$ | 5 | $1 + 4t + 3t^2$ |
| | $c^2 = 1$ | $z_0^2(z_0 - z_6)^2(z_0 + z_6)^2$ | 3 | $1 + 2t + t^2$ |
| 6 | $\mathfrak{s}_{5,1}^{1,2,b}$, $0 \leq \arg b < \pi$ | | | |
| | $b \neq 0, 1$ | $z_0(z_0 + z_6)^2(z_0 + 2z_6)(z_0 + (1-b)z_6)(z_0 + (1+b)z_6)$ | 5 | $1 + 4t + 3t^2$ |
| | $b = 0$ | $z_0(z_0 + z_6)^4(z_0 + 2z_6)$ | 3 | $1 + 2t + t^2$ |
| | $b = 1$ | $z_0^2(z_0 + z_6)^2(z_0 + 2z_6)^2$ | 3 | $1 + 2t + t^2$ |
| 6 | $\mathfrak{s}_{5,1}^{1,3,b}$, $0 \leq \arg b < \pi$ | | | |
| | $b \neq 0, 1$ | $z_0(z_0 + 2z_6)(z_0 + (1-b)z_6)^2(z_0 + (1+b)z_6)^2$ | 4 | $1 + 3t + 2t^2$ |
| | $b = 0$ | $z_0(z_0 + 2z_6)(z_0 + z_6)^4$ | 3 | $1 + 2t + t^2$ |
| | $b = 1$ | $z_0^3(z_0 + 2z_6)^3$ | 2 | $1 + t$ |
| 6 | $\mathfrak{s}_{5,1}^{2,1,b,c}$, $0 \leq \arg b, \arg c < \pi$ and $\lvert c\rvert \leq \lvert b\rvert$ | | | |
| | $b = c = 0$ | $z_0(z_0 + 2z_6)(z_0 + z_6)^4$ | 3 | $1 + 2t + t^2$ |
| | $c = 0,\ b \neq 0, 1$ | $z_0(z_0 + 2z_6)(z_0 + z_6)^2(z_0 + (1-b)z_6)(z_0 + (1+b)z_6)$ | 5 | $1 + 4t + 3t^2$ |
| | $b = c,\ b \neq 0, 1$ | $z_0(z_0 + 2z_6)(z_0 + (1-b)z_6)^2(z_0 + (1+b)z_6)^2$ | 4 | $1 + 3t + 2t^2$ |
| | $b = 1,\ c = 0$ | $z_0^2(z_0 + 2z_6)^2(z_0 + z_6)^2$ | 3 | $1 + 2t + t^2$ |
| | $b = 1,\ c \neq 0, 1$ | $z_0^2(z_0 + 2z_6)^2(z_0 + (1-c)z_6)(z_0 + (1+c)z_6)$ | 4 | $1 + 3t + 2t^2$ |

| Dim | Algebra $\mathfrak{s}$ | $Q_{\mathfrak{s}}(z)$ | $k(\mathfrak{s})$ | $P_{\mathfrak{s}}(t)$ |
|---|---|---|---|---|
| | $c = 1$, $b \neq 1$ | $z_0^2(z_0 + 2z_6)^2(z_0 + (1-b)z_6)(z_0 + (1+b)z_6)$ | 4 | $1 + 3t + 2t^2$ |
| | $b = c = 1$ | $z_0^3(z_0 + 2z_6)^3$ | 2 | $1 + t$ |
| | $b, c \notin \{0,1\}$, $b \neq c$ | $z_0(z_0 + 2z_6)(z_0 + (1-b)z_6)(z_0 + (1+b)z_6)$ $\times(z_0 + (1-c)z_6)(z_0 + (1+c)z_6)$ | 6 | $1 + 5t + 4t^2$ |
| 7 | $\mathfrak{s}_{5,2}^{0,1}$ | $z_0^3(z_0 - z_6)(z_0 + z_6)(z_0 - z_7)(z_0 + z_7)$ | 5 | $1 + 4t + 6t^2 + 3t^3$ |
| 7 | $\mathfrak{s}_{5,2}^{0,2}$ | $z_0^3(z_0 - z_6)(z_0 + z_6)(z_0 - z_7)(z_0 + z_7)$ | 5 | $1 + 4t + 6t^2 + 3t^3$ |
| 7 | $\mathfrak{s}_{5,2}^{0,3}$ | $z_0^2(z_0 + 2z_6)(z_0 + z_6 - z_7)^2(z_0 + z_6 + z_7)^2$ | 4 | $1 + 3t + 3t^2 + t^3$ |
| 7 | $\mathfrak{s}_{5,2}^{0,4}$ | $z_0^2(z_0 + 2z_6)(z_0 + z_6 - z_7)^2(z_0 + z_6 + z_7)^2$ | 4 | $1 + 3t + 3t^2 + t^3$ |
| 7 | $\mathfrak{s}_{5,2}^{0,5}$ | $z_0^2(z_0 + z_6)^2(z_0 + 2z_6)(z_0 + z_6 - z_7)(z_0 + z_6 + z_7)$ | 5 | $1 + 4t + 5t^2 + 2t^3$ |
| 7 | $\mathfrak{s}_{5,2}^{1,1,b}$, $0 \le \arg b < \pi$ | | | |
| | all allowed $b$ | $z_0^2(z_0 + z_6)^2(z_0 + 2z_6)(z_0 + (1-b)z_6 + z_7)(z_0 + (1+b)z_6 - z_7)$ | 5 | $1 + 4t + 5t^2 + 2t^3$ |
| 7 | $\mathfrak{s}_{5,2}^{1,2,b}$, $0 \le \arg b < \pi$ | | | |
| | all allowed $b$ | $z_0^2(z_0 + 2z_6)(z_0 + (1-b)z_6 - z_7)^2(z_0 + (1+b)z_6 + z_7)^2$ | 4 | $1 + 3t + 3t^2 + t^3$ |
| 7 | $\mathfrak{s}_{5,2}^{2,1,b,c}$, $0 < \arg b < \pi$ and $\lvert c\rvert \le 1$ | | | |
| | $c \in \mathcal{E}_b$ | $z_0^2(z_0 + 2z_6)(z_0 + z_6 - z_7)(z_0 + z_6 + z_7)$ $\times(z_0 + (1-b)z_6 - cz_7)(z_0 + (1+b)z_6 + cz_7)$ | 6 | $1 + 5t + 9t^2 + 5t^3$ |
| | $c \notin \mathcal{E}_b$ | $z_0^2(z_0 + 2z_6)(z_0 + z_6 - z_7)(z_0 + z_6 + z_7)$ $\times(z_0 + (1-b)z_6 - cz_7)(z_0 + (1+b)z_6 + cz_7)$ | 6 | $1 + 5t + 10t^2 + 6t^3$ |
| 8 | $\mathfrak{s}_{5,3}^{0,1}$ | $z_0^3(z_0 + 2z_6)(z_0 + z_6 - z_7)(z_0 + z_6 + z_7)$ $\times(z_0 + z_6 - z_8)(z_0 + z_6 + z_8)$ | 6 | $1 + 5t + 10t^2 +$ $9t^3 + 3t^4$ |

3.5. **Spectral Equivalence Classification.** We now describe the spectral equivalence relation on the algebras in Table 1. We use Proposition 2.7 in the following form. If

$$Q_{\mathfrak{s}} = z_0^{e_0(\mathfrak{s})} \prod_{\alpha \in \Delta_{\mathfrak{s}}^{\times}} (z_0 + \alpha)^{m_\alpha},$$

where $\Delta_{\mathfrak{s}}^{\times}$ is the multiset of nonzero nilradical weights, then $\mathfrak{s} \sim \mathfrak{s}'$ if and only if $e_0(\mathfrak{s}) = e_0(\mathfrak{s}')$ and the multisets $\Delta_{\mathfrak{s}}^{\times}$ and $\Delta_{\mathfrak{s}'}^{\times}$ are linearly equivalent.

For a one-dimensional extension space, linear equivalence means multiplication by a nonzero scalar. We record the elementary comparisons needed below.

**Lemma 3.15.** *Let $S$ and $S'$ be finite multisets of nonzero complex numbers. The associated one-dimensional nilradical weight multisets are linearly equivalent if and only if $S' = \lambda S$ as multisets for some $\lambda \in \mathbb{C}^{\times}$.*

*The following comparisons will be used below.*

(1) *If $b, b' \neq \pm 1$, then the multisets $\{2, 1-b, 1+b\}$ and $\{2, 1-b', 1+b'\}$ are linearly equivalent if and only if $b' = b$ or $b' = -b$.*
(2) *If $b, b' \neq \pm 1$, then the multisets*

$$\{2, 1, 1, 1-b, 1+b\} \quad \textit{and} \quad \{2, 1, 1, 1-b', 1+b'\}$$

*are linearly equivalent if and only if $b' = b$ or $b' = -b$.*
(3) *If $b, b' \neq \pm 1$, then the multisets*

$$\{2, 1-b, 1-b, 1+b, 1+b\} \quad \textit{and} \quad \{2, 1-b', 1-b', 1+b', 1+b'\}$$

*are linearly equivalent if and only if $b' = b$ or $b' = -b$.*

(4) *If the five numbers* $2, 1-b, 1+b, 1-c, 1+c$ *are nonzero and pairwise distinct, then the multisets*

$$\{2, 1-b, 1+b, 1-c, 1+c\} \quad \text{and} \quad \{2, 1-b', 1+b', 1-c', 1+c'\}$$

*are linearly equivalent if and only if*

$$\{\pm b, \pm c\} = \{\pm b', \pm c'\}.$$

(5) *If* $c, c' \neq 0$, *then the multisets* $\{1, -1, c, -c\}$ *and* $\{1, -1, c', -c'\}$ *are linearly equivalent if and only if*

$$c' \in \{\pm c, \pm c^{-1}\}.$$

*After imposing a chosen normal range, only the values in this set which remain in that range are retained.*

*Proof.* A linear automorphism of a one-dimensional vector space acts on the dual space by multiplication by a nonzero scalar. This proves the first claim.

In the first four comparisons, the sums of the displayed multisets are nonzero. Thus, if $S' = \lambda S$, comparison of total sums gives $\lambda = 1$. The stated parameter identifications then follow by comparing the multisets directly. In the last comparison, the total sum is zero. A common nonzero scaling may normalize either pair to $\{\pm 1\}$, and the remaining pair is then determined up to sign and inversion. □

It remains to treat the two-dimensional family $\mathfrak{s}_{5,2}^{2,1,b,c}$. Let $x, y$ be the dual basis corresponding to $z_6, z_7$. The nonzero nilradical weights are

$$2x, \qquad x-y, \qquad x+y, \qquad (1-b)x - cy, \qquad (1+b)x + cy.$$

Set

$$\gamma = 2x, \qquad A = x - y, \qquad B = x + y, \qquad C = (1-b)x - cy, \qquad D = (1+b)x + cy,$$

and write

$$W(b, c) = \{\gamma, A, B, C, D\}$$

as a multiset. Then $A + B = \gamma$ and $C + D = \gamma$. We again write

$$\mathcal{E}_b = \{0,\ b+1,\ b-1,\ 1-b,\ -1-b\}.$$

The values $c \in \mathcal{E}_b$ are separated from the regular values $c \notin \mathcal{E}_b$ by the spectral matroid, because they are the values for which some three forms in

$$\{z_0 + \gamma,\ z_0 + A,\ z_0 + B,\ z_0 + C,\ z_0 + D\}$$

are linearly dependent.

**Lemma 3.16.** *Consider* $\mathfrak{s}_{5,2}^{2,1,b,c}$ *with* $0 < \arg b < \pi$ *and* $|c| \leq 1$.

*If* $c \notin \mathcal{E}_b$ *and* $c' \notin \mathcal{E}_{b'}$, *then* $\mathfrak{s}_{5,2}^{2,1,b,c} \sim \mathfrak{s}_{5,2}^{2,1,b',c'}$ *if and only if* $(b', c')$ *is obtained from* $(b, c)$ *by the operations generated by*

$$(b, c) \mapsto (b, -c) \qquad \text{and} \qquad (b, c) \mapsto (-b/c, 1/c),$$

*with the requirement that the resulting pair lies in the parameter range.*

*If* $c \in \mathcal{E}_b$ *and* $c' \notin \mathcal{E}_{b'}$, *then the two algebras are not spectrally equivalent. If* $c, c'$ *are both exceptional, then* $\mathfrak{s}_{5,2}^{2,1,b,c} \sim \mathfrak{s}_{5,2}^{2,1,b',c'}$ *if and only if the multisets* $W(b, c)$ *and* $W(b', c')$ *are linearly equivalent.*

*Proof.* The distinction between $c \in \mathcal{E}_b$ and $c \notin \mathcal{E}_b$ is preserved by spectral equivalence, since spectral equivalence preserves the spectral matroid.

Assume first that $c \notin \mathcal{E}_b$ and $c' \notin \mathcal{E}_{b'}$. In the regular case, the five forms $z_0 + \lambda$ with $\lambda \in W(b, c)$ have no three-element dependent subset. The element $\gamma$ is then distinguished by the two decompositions

$$A + B = \gamma, \qquad C + D = \gamma.$$

Thus any linear equivalence between two regular configurations must preserve the corresponding element $\gamma$ and must either preserve or interchange the two unordered pairs $\{A, B\}$ and $\{C, D\}$.

Interchanging $A$ and $B$ sends $y$ to $-y$, and therefore sends $(b,c)$ to $(b,-c)$. Interchanging the two decompositions replaces the role of $y$ by $bx+cy$. Rewriting the old basis in terms of the new one gives

$$(b,c)\mapsto(-b/c,1/c).$$

These operations generate the stated identifications, subject to the requirement that the resulting pair remain in the chosen normal range.

If both parameter values are exceptional, then the preceding regular argument is not used. In that case Proposition 2.7 applies directly and gives the criterion that $W(b,c)$ and $W(b',c')$ be linearly equivalent. □

**Theorem 3.17.** *The spectral equivalence relation on the algebras in Table 1 is given in Table 2.*

Table 2: Spectral equivalence classes for low-dimensional solvable Lie algebras with Heisenberg nilradical

| Dim | Spectral equivalence classes and rules |
|---|---|
| 4 | The algebra $\mathfrak{s}_{3,1}^{0,1}$ is a singleton spectral equivalence class. Also $$\mathfrak{s}_{3,1}^{0,2}\sim\mathfrak{s}_{3,1}^{1,1,0}.$$ The algebra $\mathfrak{s}_{3,1}^{1,1,1}$ is a singleton spectral equivalence class. For $b>0$ and $b\neq 1$, the algebras $\mathfrak{s}_{3,1}^{1,1,b}$ are pairwise spectrally inequivalent. |
| 5 | The algebra $\mathfrak{s}_{3,2}^{0,1}$ is the only dimension-5 algebra in the table. |
| 6 | The following algebras form one spectral equivalence class: $$\mathfrak{s}_{5,1}^{0,1}\sim\mathfrak{s}_{5,1}^{0,4}\sim\mathfrak{s}_{5,1}^{1,2,0}\sim\mathfrak{s}_{5,1}^{1,3,0}\sim\mathfrak{s}_{5,1}^{2,1,0,0}.$$ The next two equivalences are $$\mathfrak{s}_{5,1}^{0,2}\sim\mathfrak{s}_{5,1}^{1,1,0},\qquad \mathfrak{s}_{5,1}^{0,3}\sim\mathfrak{s}_{5,1}^{1,1,c}\quad (c^2=1).$$ For $0<\lvert c\rvert\le 1$ with $c^2\neq 1$, the algebras $\mathfrak{s}_{5,1}^{1,1,c}$ are compared by the orbit of $\{1,-1,c,-c\}$ under common nonzero scaling; equivalently, $c'\in\{\pm c,\pm c^{-1}\}$, subject to the chosen normal range. For every allowed $b$, $$\mathfrak{s}_{5,1}^{1,2,b}\sim\mathfrak{s}_{5,1}^{2,1,b,0},\qquad \mathfrak{s}_{5,1}^{1,3,b}\sim\mathfrak{s}_{5,1}^{2,1,b,b}.$$ If $b\neq 0,\pm1$, then $b$ is determined up to sign, whenever both values lie in the normal range. The remaining members of $\mathfrak{s}_{5,1}^{2,1,b,c}$ are compared by the nonzero weight multiset $$\{2,1-b,1+b,1-c,1+c\}\setminus\{0\},$$ together with the total multiplicity of the factor $z_0$. If the five displayed weights are nonzero and pairwise distinct, this reduces to equality of the unordered signed pair $\{\pm b,\pm c\}$. |
| 7 | The following equivalences hold: $$\mathfrak{s}_{5,2}^{0,1}\sim\mathfrak{s}_{5,2}^{0,2},$$ and, for every allowed $b$, $$\mathfrak{s}_{5,2}^{0,3}\sim\mathfrak{s}_{5,2}^{0,4}\sim\mathfrak{s}_{5,2}^{1,2,b},\qquad \mathfrak{s}_{5,2}^{0,5}\sim\mathfrak{s}_{5,2}^{1,1,b}.$$ No member of the family $\mathfrak{s}_{5,2}^{2,1,b,c}$ is spectrally equivalent to any of the preceding dimension-7 algebras in the table. Inside this family, write $$\mathcal{E}_b=\{0,\ b+1,\ b-1,\ 1-b,\ -1-b\}$$ and $$W(b,c)=\{2x,\ x-y,\ x+y,\ (1-b)x-cy,\ (1+b)x+cy\}.$$ If $c\notin\mathcal{E}_b$ and $c'\notin\mathcal{E}_{b'}$, then equivalence is generated by $$(b,c)\mapsto(b,-c),\qquad (b,c)\mapsto(-b/c,1/c),$$ subject to the parameter range. If both parameter values are exceptional, equivalence is given by linear equivalence of the multisets $W(b,c)$ and $W(b',c')$. Regular and exceptional parameter values are not spectrally equivalent. |
| 8 | The algebra $\mathfrak{s}_{5,3}^{0,1}$ is the only dimension-8 algebra in the table. |

*Proof.* We compare $e_0(\mathfrak{s})$ and the nonzero nilradical weight multiset $\Delta_{\mathfrak{s}}^{\times}$.

In dimension 4, the nonzero weight multisets are

$$\{-1,1\}, \qquad \{1,1,2\}, \qquad \{2,1-b,1+b\}\setminus\{0\},$$

for $\mathfrak{s}_{3,1}^{0,1}$, $\mathfrak{s}_{3,1}^{0,2}$, and $\mathfrak{s}_{3,1}^{1,1,b}$, respectively. The total $z_0$-multiplicity separates the cases not listed as equivalent, and Lemma 3.15 gives the remaining dimension-4 claims.

The dimension-5 case contains only $\mathfrak{s}_{3,2}^{0,1}$.

In dimension 6, all extension spaces are one-dimensional. The relevant nonzero weight multisets are

$$\{2,1,1,1,1\}, \qquad \{-1,1\}, \qquad \{-1,-1,1,1\},$$

for the non-parameter classes appearing in Table 2. The parameter families give

$$\{-1,1,-c,c\}\setminus\{0\},$$
$$\{2,1,1,1-b,1+b\}\setminus\{0\},$$
$$\{2,1-b,1-b,1+b,1+b\}\setminus\{0\},$$

and

$$\{2,1-b,1+b,1-c,1+c\}\setminus\{0\}.$$

Applying Lemma 3.15, together with the total multiplicity of the factor $z_0$, gives the dimension-6 claims.

For dimension 7, the first two algebras have the same nonzero weight configuration $\{\pm x,\pm y\}$. The algebras $\mathfrak{s}_{5,2}^{0,3}$, $\mathfrak{s}_{5,2}^{0,4}$, and $\mathfrak{s}_{5,2}^{1,2,b}$ have nonzero weight multiset equivalent to

$$\{2x, x-y, x+y\},$$

with multiplicities, after a change of basis in the extension space. Similarly, $\mathfrak{s}_{5,2}^{0,5}$ and $\mathfrak{s}_{5,2}^{1,1,b}$ have nonzero weight multiset equivalent to

$$\{x, x, 2x, x-y, x+y\}$$

after a change of basis. The family $\mathfrak{s}_{5,2}^{2,1,b,c}$ has five distinct non-$z_0$ factors in the table, so it is separated from the preceding dimension-7 classes by $k(\mathfrak{s})$. Its internal comparison is Lemma 3.16.

The dimension-8 part contains only $\mathfrak{s}_{5,3}^{0,1}$. This proves the theorem. □

## References


[ABCSGV11] J. M. Ancochea Bermudez, R. Campoamor-Stursberg, and L. Garcia Vergnolle. Classification of lie algebras with naturally graded quasi-filiform nilradicals. *Journal of Geometry and Physics*, 61:2168–2186, 2011.

[ACSGV06] J. M. Ancochea, R. Campoamor-Stursberg, and L. Garcia Vergnolle. Solvable lie algebras with naturally graded nilradicals and their invariants. *J. Phys. A: Math. Gen.*, 39:1339–1355, 2006.

[Car04] Élie Cartan. Sur la structure des groupes infinis de transformation. *Annales scientifiques de l'École normale supérieure*, 21:153–206, 1904.

[CSZ15] Isaak Chagouel, Michael Stessin, and Kehe Zhu. Geometric spectral theory for compact operators. *Transactions of the American Mathematical Society*, 368(3):1559–1582, 2015.

[Cur92] Charles W. Curtis. Representation theory of finite groups: From frobenius to brauer. *The Mathematical Intelligencer*, 14(4):48–57, 1992.

[Cur99] Charles Curtis. *Pioneers of Representation Theory: Frobenius, Burnside, Schur, and Brauer*. American Mathematical Society, October 1999.

[CY13] Patrick Cade and Rongwei Yang. Projective spectrum and cyclic cohomology. *Journal of Functional Analysis*, 265(9):1916–1933, November 2013.

[Dic02] Leonard Eugene Dickson. An elementary exposition of frobenius's theory of group-characters and group-determinants. *The Annals of Mathematics*, 4(1):25, October 1902.

[Dic21] Leonard Eugene Dickson. Determination of all general homogeneous polynomials expressible as determinants with linear elements. *Transactions of the American Mathematical Society*, 22(2):167–179, 1921.

[DY18] Ronald G. Douglas and Rongwei Yang. *Hermitian Geometry on Resolvent Set*, page 167–183. Springer International Publishing, 2018.

[FLW26] Chenyue Feng, Shoumin Liu, and Xumin Wang. Characteristic polynomials for classical lie algebras and their orbit decompositions. *Linear Algebra and its Applications*, 734:28–49, April 2026.

[Fol89] Gerald B. Folland. *Harmonic Analysis in Phase Space*, volume 122 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, March 1989.

[Fro96] G. Frobenius. Über vertauschbare matrizen. *Sitzungsberichte der Königlich Preussischen Akademie der Wissenschaften zu Berlin*, pages 601–604, 1896.

[FS91] Edward Formanek and David Sibley. The group determinant determines the group. *Proceedings of the American Mathematical Society*, 112(3):649–656, 1991.

[GGRW89] L Gagnon, B Grammaticos, A Ramani, and P Winternitz. Lie symmetries of a generalised non-linear schrodinger equation. iii. reductions to third-order ordinary differential equations. *Journal of Physics A: Mathematical and General*, 22(5):499–509, March 1989.

[GH94] Michel Goze and You Hakimjanov. Sur les algèbres de lie nilpotentes admettant un tore de dérivations. *Manuscripta Mathematica*, 84(1):115–124, December 1994.

[GLW24] Amin Geng, Shoumin Liu, and Xumin Wang. Characteristic polynomials and finite dimensional representations of simple lie algebras. *New York Journal of Mathematics*, 30:24–37, January 2024.

[Gon98] Ming-Peng Gong. *Classification of Nilpotent Lie Algebras of Dimension 7 (Over Algebraically Closed Fields and R)*. Phd thesis, University of Waterloo, Waterloo, Ontario, Canada, 1998.

[Goo70] R. L. Goodstein. Richard dedekind. gesammelte mathematische werke. 2 volumes. edited by r. fricke, emmy noether and oystein ore. 1969. (chelsea, new york.) - vorlesungen über zahlentheorie. by p. g. lejeune dirichlet. supplemented and revised by r. dedekind. 1968. (chelsea, new york.). *The Mathematical Gazette*, 54(388):173–173, May 1970.

[GY17] Rostislav Grigorchuk and Rongwei Yang. Joint spectrum and the infinite dihedral group. *Proceedings of the Steklov Institute of Mathematics*, 297(1):145–178, May 2017.

[GY22] Bryan Goldberg and Rongwei Yang. Self-similarity and spectral dynamics. *Journal of Operator Theory*, 87(1):355–388, March 2022.

[Heb98] Jens Heber. Noncompact homogeneous einstein spaces. *Inventiones Mathematicae*, 133(2):279–352, 1998.

[HK12] June Huh and Eric Katz. Log-concavity of characteristic polynomials and the bergman fan of matroids. *Mathematische Annalen*, 354(3):1103–1116, January 2012.

[HT08] Firas Hindeleh and Gerard Thompson. Seven dimensional lie algebras with a four-dimensional nilradical. *Algebras, Groups and Geometries*, 25(3):243, 2008.

[HWY17] Wei He, Xiaofeng Wang, and Rongwei Yang. Projective spectrum and kernel bundle. ii. *Journal of Operator Theory*, 78(2):417–433, October 2017.

[HZ19] Zhiguang Hu and Philip B. Zhang. Determinants and characteristic polynomials of lie algebras. *Linear Algebra and its Applications*, 563:426–439, February 2019.

[Kha89] Yu. B. Khakimdzhanov. Characteristically nilpotent lie algebras. *Algebra and Logic*, 28(6):475–485, November 1989.

[Kil88] Killing. Die zusammensetzung der stetigen endlichen transformationsgruppen. *Mathematische Annalen*, 31:252–290, 1888.

[Kir04] A. Kirillov. *Lectures on the Orbit Method*. American Mathematical Society, 2004.

[KY21] Fatemeh Azari Key and Rongwei Yang. Spectral invariants for finite dimensional lie algebras. *Linear Algebra and its Applications*, 611:148–170, 2021.

[Lau10] Jorge Lauret. Einstein solvmanifolds are standard. *Annals of Mathematics*, 172(3):1859–1877, October 2010.

[Lev05] Eugenio Elia Levi. Sulla struttura dei gruppi finiti e continui. *Atti della Reale Accademia delle Scienze di Torino*, 40:551–565, 1905.

[LNN+22] Vu A. Le, Tuan A. Nguyen, Tu T. C. Nguyen, Tuyen T. M. Nguyen, and Thieu N. Vo. Classification of 7-dimensional solvable lie algebras having 5-dimensional nilradicals. *Communications in Algebra*, 51(5):1866–1885, December 2022.

[Mal45] A. Malcev. On solvable lie algebras. *Izvestiya Akademii Nauk SSSR. Seriya Matematicheskaya*, 9(5):329–356, 1945.

[Mor58] V. V. Morozov. Classification of nilpotent lie algebras of sixth order. *Izvestiya Vysshikh Uchebnykh Zavedenii. Matematika*, (4):161–171, 1958. MathSciNet: https://mathscinet.ams.org/mathscinet-getitem?mr=130326; zbMATH: https://zbmath.org/?q=an:0198.05501.

[MQW17] Tong Mao, Yikun Qiao, and Penghui Wang. Commutativity of normal compact operators via projective spectrum. *Proceedings of the American Mathematical Society*, 146(3):1165–1172, 2017.

[Mub63a] G. M. Mubarakzyanov. Classification of real structures of lie algebras of fifth order. *Izvestiya Vysshikh Uchebnykh Zavedenii. Matematika*, (3):99–106, 1963.

[Mub63b] G. M. Mubarakzyanov. Classification of solvable lie algebras of sixth order with a non-nilpotent basis element. *Izvestiya Vysshikh Uchebnykh Zavedenii. Matematika*, (4):104–116, 1963.

[Mub63c] G. M. Mubarakzyanov. On solvable lie algebras. *Izvestiya Vysshikh Uchebnykh Zavedenii. Matematika*, (1):114–123, 1963.

[Mul23] Michael Muller. *A Spectral Classification Of Solvable Lie Algebras*. PhD thesis, University Libraries, University at Albany, State University of New York, 2023.

[NW94] J C Ndogmo and P Winternitz. Solvable lie algebras with abelian nilradicals. *Journal of Physics A: Mathematical and General*, 27(2):405–423, January 1994.

[OS80] Peter Orlik and Louis Solomon. Combinatorics and topology of complements of hyperplanes. *Inventiones Mathematicae*, 56(2):167–189, February 1980.

[Par13] Alan R. Parry. A classification of real indecomposable solvable lie algebras of small dimension with codimension one nilradicals. Master's thesis, 2013.

[PSWZ76] J. Patera, R. T. Sharp, P. Winternitz, and H. Zassenhaus. Invariants of real low dimension lie algebras. *Journal of Mathematical Physics*, 17(6):986–994, 1976.

[RR77] G. Rosensteel and D. J. Rowe. Nuclear¡mml:math xmlns:mml="http://www.w3.org/1998/math/mathml" display="inline"¿¡mml:mi mathvariant="normal"¿sp¡/mml:mi¿¡mml:mo¿(¡/mml:mo¿¡mml:mn¿3¡/mml:mn¿¡mml:mo¿, ¡/mml:mo¿¡mml:mi¿ ¡/mml:mi¿¡mml:mi¿r¡/mml:mi¿¡mml:mo¿)¡/mml:mo¿¡/mml:math¿model. *Physical Review Letters*, 38(1):10–14, January 1977.

[RW93] J L Rubin and P Winternitz. Solvable lie algebras with heisenberg ideals. *Journal of Physics A: Mathematical and General*, 26(5):1123–1138, March 1993.

[See93] Craig Seeley. 7-dimensional nilpotent lie algebras. *Transactions of the American Mathematical Society*, 335(2):479, February 1993.

[SK10] L. Snobl and D. Karásek. Classification of solvable lie algebras with a given nilradical by means of solvable extensions of its subalgebras. *Linear Algebra and its Applications*, 432(7):1836–1850, March 2010.

[Sno11] Libor Snobl. Maximal solvable extensions of filiform algebras. *Archivum Mathematicum*, 047(5):405–414, 2011.

[SW05] L Snobl and P Winternitz. A class of solvable lie algebras and their casimir invariants. *Journal of Physics A: Mathematical and General*, 38(12):2687–2700, March 2005.

[SW09] L Snobl and P Winternitz. All solvable extensions of a class of nilpotent lie algebras of dimension¡i¿n¡/i¿and degree of nilpotency¡i¿n¡/i¿ 1. *Journal of Physics A: Mathematical and Theoretical*, 42(10):105201, February 2009.

[SW12] L Snobl and P Winternitz. Solvable lie algebras with borel nilradicals. *Journal of Physics A: Mathematical and Theoretical*, 45(9):095202, February 2012.

[SW14] Libor Snobl and Pavel Winternitz. *Classification and Identification of Lie Algebras*. American Mathematical Society, 2014.

[SYZ11] Michael Stessin, Rongwei Yang, and Kehe Zhu. Analyticity of a joint spectrum and a multivariable analytic fredholm theorem. *New York Journal of Mathematics*, 17a:39–44, 2011.

[TKK00] GR. Tsagas, A. Kobotis, and T. Koukouvinos. Classification of nilpotent lie algebras of dimension nine whose maximum abelian ideal is of dimension seven. *International Journal of Computer Mathematics*, 74(1):5–28, January 2000.

[Tur90] P. Turkowski. Solvable lie algebras of dimension six. *Journal of Mathematical Physics*, 31(6):1344–1350, 1990.

[TW98] S Tremblay and P Winternitz. Solvable lie algebras with triangular nilradicals. *Journal of Physics A: Mathematical and General*, 31(2):789–806, January 1998.

[WLD08] Yan Wang, Jie Lin, and Shaoqiang Deng. Solvable lie algebras with quasifiliform nilradicals. *Communications in Algebra*, 36(11):4052–4067, November 2008.

[Woi17] Peter Woit. *Quantum Theory, Groups and Representations*. Springer International Publishing, 2017.

[Yan09] Rongwei Yang. Projective spectrum in banach algebras. *Journal of Topology and Analysis*, 01(03):289–306, 2009.

[Yan24] Rongwei Yang. *A Spectral Theory Of Noncommuting Operators*. Springer Nature Switzerland, 2024.

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