G^∞ DeWitt Supermanifolds

Gary Hu

November 2024

Abstract

This expository paper is an introduction to supergeometry through the study of G^∞-DeWitt supermanifolds. After briefly discussing the physical significance of the theory and its relevance, we delve into the fundamentals of supermathematics, gradually progressing to defining the $\mathbb{R}_S^{m,n}$ superspace and the associated DeWitt topology. These building blocks allow us to introduce the G^∞-DeWitt supermanifolds, which are the supergeometric analogs of smooth manifolds. Then, we present a concrete example: the real super projective space $\mathbb{SRP}^{m,n}$. Finally, we conclude with a brief overview of modern supergeometry.

Contents

1 The Big Picture of Supermathematics

1.1 Why Supersymmetry?

Over the past 50 years, the Standard Model of particle physics has become one of the most successful theories, classifying all known elementary particles and describing three of the four fundamental forces. However, a major question remains: How can we unify all four fundamental forces under a single theory?

Supersymmetry was proposed as an extension of the Standard Model to help address this issue. It suggests a symmetry between fermions (half-integer spin particles like quarks and leptons) and bosons (integer spin particles like photons and gluons). Supersymmetry predicts that each particle has a superpartner whose spin differs by $\frac{1}{2}$: fermions pair with bosonic superpartners, and bosons pair with fermionic ones. For example, the electron is paired with a selectron, and the photon is paired with a photino.

Theoretically, supersymmetry resolves several important issues:

- The Hierarchy Problem: Why is the weak force scale (related to the mass of the Higgs boson) 10^{16} smaller than the Planck scale (the scale where quantum gravity becomes important)? This difference in scales creates a problem because quantum corrections grow quadratically with energy, leading to an unnaturally large mass for the Higgs Boson. Supersymmetry addresses this by introducing superpartners for each Standard Model particle. In supersymmetry, the quantum corrections from fermions and bosons have opposite signs, leading to a cancellation of large corrections.

- Unification of Forces: In the Standard Model, the three gauge couplings - strong, weak, and electromagnetic - evolve with energy but do not converge at any scale. Supersymmetry improves this by introducing superpartners, which modify the coupling constants. In supersymmetry models, the contributions from both Standard Model particles and their superpartners cause the couplings to evolve in such a way that they converge at a single energy scale, around 10^{16} GeV, known as the Grand Unified Theory scale. This convergence suggests a potential unification of the forces at high energy, a feature of supersymmetry models like the Minimal Supersymmetric Standard Model.

- Dark Matter: Although invisible, dark matter is widely believed to exist because of its gravitational effects on visible matter, such as the rotation curves of galaxies and patterns in the cosmic microwave background. While it has not been detected experimentally, it is believed to consist of particles that interact only very weakly with normal matter. In the context of supersymmetry, a promising dark matter candidate is the lightest supersymmetric particle, typically a neutralino, a mixture of photino, zino, higgsino, and the bino.

- String Theory: String theory is a framework that attempts to unify all fundamental forces, including gravity, by describing particles as one-dimensional "strings" rather than point-like particles. Supersymmetry is a critical ingredient to make string theory consistent and mathematically well-defined. Without supersymmetry, string theory would face inconsistencies, such as the absence of a consistent quantum theory of gravity. In particular, supersymmetry ensures that the theory includes a consistent massless spin-2 particle (the graviton), which is required to describe gravity in quantum mechanics. Consequently, superstring theory provides a potential framework for unifying all fundamental forces.

Despite its promising implications, supersymmetry has not been confirmed experimentally. Extensive searches at the Large Hadron Collider have failed to detect any superpartners. Although many physicists now consider supersymmetry unlikely to be a complete theory, suggesting instead that it might describe a different universe rather than our own, the principles of supersymmetry continue to offer valuable insights into the possible structure of the universe. As a result, the mathematics behind supersymmetry remains an active area of research.

1.2 Big Themes in Supermathematics

The mathematics underlying supersymmetry is known as supermathematics. One of the fascinating aspects of supermathematics is that many mathematical structures and theorems can be made "super". Consequently, the development of supermathematics has largely involved creating super-analogs for existing mathematical objects and results.

There are two big recurring themes throughout all of supermathematics:

- Grading by parity: All objects in supermathematics are graded modulo 2, meaning they are classified as either "even" or "odd." Even objects correspond to bosons, which obey standard commutation relations, while odd objects correspond to fermions, which follow anti-commutation relations.
- Anticommutativity of odd elements: In classical formulas, swapping the order of two odd objects results in a sign change. This property reflects the Pauli exclusion principle, which states that no two identical fermions can occupy the same quantum state. Mathematically, the Pauli exclusion leads to anticommutativity, explaining the appearance of minus signs when switching the order of odd elements.

For this paper, our primary focus is on supergeometry, which extends the concepts of differential geometry into supermathematics. Specifically, we aim to explore how smooth manifolds can be turned into supermanifolds.

1.3 Two Approaches To Supermanifolds

There are two (equivalent) approaches to defining supermanifolds:

- The concrete approach: This approach treats a supermanifold as a manifold that is locally modeled on a flat "superspace." The coordinates of this superspace include both even and odd components from the Grassmann algebra.
- The algebro-geometric approach: This approach defines a supermanifold by extending the sheaf of functions on a manifold.

Each of these approaches has its strengths and weaknesses. The concrete approach is generally favored by physicists because it is easier to work with and draws many parallels to classical differential geometry. Meanwhile, the algebro-geometric approach is more abstract, mathematically elegant, and appealing to mathematicians. As both approaches are equivalent, it is generally a good rule to pick between the two based on specific problems.

In this expository paper, we focus on the concrete approach.

2 The m, n-dim Flat Superspace

In classical differential geometry, we want our manifolds to look locally like Euclidean space. In supergeometry, we want our manifolds to look like flat superspace instead. But before delving into the structure of superspace, we must first introduce some fundamental concepts in the context of supergeometry.

2.1 Super Algebras

Perhaps the most basic and well-studied object in mathematics is the vector space. This serves as the foundation for more abstract structures, and naturally, a good starting point is to generalize this concept to incorporate supersymmetry. We begin by introducing a *super vector space*, which are vector spaces with a $\mathbb{Z}_2$ grading:

Definition 2.1. *A* ***super vector space*** $\mathbb{V}$ *is a vector space, together with a choice of two subspaces* $\mathbb{V}_0$ *and* $\mathbb{V}_1$ *of* $\mathbb{V}$ *such that*

$$\mathbb{V} = \mathbb{V}_0 \oplus \mathbb{V}_1.$$

Elements of the subspace $\mathbb{V}_0$ *are* ***even*** *and elements of* $\mathbb{V}_1$ *are said to be* ***odd****. An element* V *which belongs to* $\mathbb{V}_i$ *where* $i = 0$ *or* 1 *is said to be* ***homogeneous*** *and the* $\mathbb{Z}_2$ ***degree*** *of* V *is* $|V| = i$.

Once we have the notion of a vector space, it is natural to ask about the next level of structure: an algebra. Let's do this:

Definition 2.2. *Suppose that* $\mathbb{A}$ *is an algebra over* $\mathbb{R}$ *or* $\mathbb{C}$. *Then* $\mathbb{A}$ *is a* ***super algebra*** *if it is a super vector space (over the same field) and*

$$\mathbb{A}_i \mathbb{A}_j \subset \mathbb{A}_{i+j} \qquad i, j = 0, 1.$$

where addition is mod 2.

From the underlying physics, bosons lead to commutativity, while fermions lead to anticommutativity. With this in mind, it's straightforward to infer the definition of super commutativity:

Definition 2.3. *The super algebra* $\mathbb{A}$ *is* ***super commutative*** *if for all* $A, B \in \mathbb{A}$,

$$AB = (-1)^{|A|\,|B|} BA.$$

2.2 Real Grassmann Algebras $\mathbb{R}_{S[L]}$

Remember that the goal of this section is to define the most important structure in supergeometry: superspace. Before formally defining superspace, let us introduce the Grassmann algebra, the super commutative algebras that are used to construct concrete supermanifolds.

Definition 2.4. *For each finite positive integer* L, $\mathbb{R}_{S[L]}$ *denotes the* ***Grassmann algebra*** *over* $\mathbb{R}$ *with* L *generators. That is,* $\mathbb{R}_{S[L]}$ *is the algebra over* $\mathbb{R}$ *with generators*

$$1, \beta_{[1]}, ..., \beta[L]$$

and relations

$$1\beta_{[i]} = \beta_{[i]} = \beta_{[i]}1 \qquad i = 1, ..., L$$
$$\beta_{[i]}\beta_{[j]} = -\beta_{[j]}\beta_{[i]} \qquad i, j = 1, ..., L$$

Theorem 2.5. *The Grassmann algebra is a super commutative algebra.*

Proof. Define

$$\mathbb{R}_{[L\,0]} = \left\{ x \mid x \in \mathbb{R}_{S[L]},\, x = \sum_{\underline{\lambda} \in M_{L\,0}} x_{\underline{\lambda}} \beta_{[\underline{\lambda}]} \right\}$$
$$\mathbb{R}_{[L\,1]} = \left\{ \xi \mid \xi \in \mathbb{R}_{S[L]},\, \xi = \sum_{\underline{\lambda} \in M_{L\,1}} \xi_{\underline{\lambda}} \beta_{[\underline{\lambda}]} \right\}$$

where $\underline{\lambda}$ is a multi index $\underline{\lambda} = \lambda_1 ... \lambda_k$ with $1 \leq \lambda_1 < ... < \lambda_k \leq L$, M_L is the set of all such multi indices (including the empty index $\emptyset$), each $X_{\underline{\lambda}}$ for $\underline{\lambda} \in M_L$ is a real number, and $\beta_{[\underline{\lambda}]} = \beta_{[\lambda_1]} ... \beta_{[\lambda_k]}$ (with $\beta_{[\emptyset]} = 1$). Now, the Grassmann algebra can be rewritten as

$$\mathbb{R}_{S[L]} = \mathbb{R}_{S[L\,0]} \oplus \mathbb{R}_{S[L\,1]}$$

so it is a super commutative algebra. □

Finally, we can describe the superspace that we wish to locally model our manifolds as:

Definition 2.6. *The* (m, n)***-dimensional flat superspace*** *is*

$$\mathbb{R}^{m,n}_{S[L]} = \underbrace{\mathbb{R}_{S[L\,0]} \times \cdots \times \mathbb{R}_{S[L\,0]}}_{m \text{ copies}} \times \underbrace{\mathbb{R}_{S[L\,1]} \times \cdots \times \mathbb{R}_{S[L\,1]}}_{n \text{ copies}}$$

with elements denoted $(x^1, ..., x^m, \xi^1, ..., \xi^n)$.

There is a crucial map, called the body map, that will be the building block for the DeWitt topology. We will state it here, but the use case will only become clear in the next subsection: the inverse of this map defines an open set in the DeWitt topology.

Theorem 2.7. *There is a unique algebra homomorphism, called the* ***body map*** *ϵ, of $\mathbb{R}_{S[L]}$ onto $\mathbb{R}$ which maps the identity element 1 onto 1 and all of the generators $\beta_{[i]}, i = 1, ..., L$ to 0 given by*

$$\begin{aligned} \epsilon : \mathbb{R}_{S[L]} &\to \mathbb{R} \\ \sum_{\underline{\lambda} \in M_L} X_{\underline{\lambda}} \beta_{[\underline{\lambda}]} &\mapsto X_{\emptyset}. \end{aligned}$$

Let's explicitly write out ϵ on our m, n-dim flat superspace:

Definition 2.8.

$$\begin{aligned} \epsilon_{m,n} : \mathbb{R}^{m,n}_{S[L]} &\to \mathbb{R}^m \\ (x^1, ..., x^m; \xi^1, ..., \xi^n) &\mapsto (\epsilon(x^1), ..., \epsilon(x^n)) \end{aligned}$$

2.3 DeWitt Topology on $\mathbb{R}^{m,n}_S$

For our purpose of defining the DeWitt topology, we mostly care about the following Grassmann algebra, which is modified to have an infinite number of generators:

Definition 2.9. *$\mathbb{R}_S$ is the algebra over $\mathbb{R}$ with generators*

$$1, \beta_{[1]}, ...$$

and relations

$$\begin{aligned} 1\beta_{[i]} = \beta_{[i]} = \beta_{[i]}1 \qquad & i = 1, 2, ... \\ \beta_{[i]}\beta_{[j]} = -\beta_{[j]}\beta_{[i]} \qquad & i, j = 1, 2, ... \end{aligned}$$

By simply replacing $\mathbb{R}_{S[L]}$ with $\mathbb{R}_S$, everything in the previous subsection holds: $\mathbb{R}_S$ is a super commutative algebra, we can construct the (m, n)-dim flat superspace $\mathbb{R}^{m,n}_S$, and the body map is the same. Most importantly, the body map on the superspace $\epsilon_{m,n}$ is identical to the previous case, except that we replace $\mathbb{R}^{m,n}_{S[L]}$ with $\mathbb{R}^{m,n}_S$.

Finally, we can define a topology on our superspace, which makes the importance of the body map clear:

Definition 2.10. *A subset U of $\mathbb{R}^{m,n}_S$ is said to be open in the* ***DeWitt topology on*** *$\mathbb{R}^{m,n}_S$ if and only if there exists an open subset V of $\mathbb{R}^m$ such that*

$$U = \epsilon^{-1}_{m,n}(V)$$

This is the best topology to use on our superspace for our purposes (it helps avoid convergence issues in infinite sums, and has algebraic structures to make other important features such as partitions of unity work). We won't go more into depth about this, rather just accepting that this is the best topology to use.

However, many other topologies can be placed on $\mathbb{R}^{m,n}_S$ by turning our superspace into a Banach or Fréchet space, but this is unnecessary for our goals.

3 G^∞-DeWitt Supermanifolds

In classical differential geometry, the most fundamental class of manifolds is smooth manifolds because we can perform calculus on them. Similarly, in supergeometry, we want to study supermanifolds that support supercalculus. In this section, we will explore one way to define supermanifolds - the G^∞ DeWitt supermanifold.

To do super calculus, we need super functions. But what exactly is a function on a (m, n)-dim flat superspace?

Definition 3.1. *A* ***superfunction*** *on an* (m, n)*-dimensional superspace takes the form*

$$f(x^1, ..., x^m; \xi^1, ..., \xi^n) = \sum_{\underline{\mu} \in M_n} f_{\underline{\mu}}(x) \xi^{\underline{\mu}} \tag{1}$$

where the $\underline{\mu}$ *are the multi indices in the set* M_n*.*

Earlier, in the case of $\mathbb{R}_{S[L]}$, we defined the multi index $\underline{\lambda} = \lambda_1 .... \lambda_k$ with $1 \leq \lambda_1 < ... < \lambda_k \leq L$, and let M_L be the set of all such multi indices. In this definition, M_n is the same as M_L, but for $\mathbb{R}_S$.

3.1 G^∞ Functions on $\mathbb{R}^{m,n}_{S[L]}$

Just as we have C^∞ functions in classical differential geometry, we seek an analogous concept for supergeometry: G^∞ functions. First, we define G^∞ functions on $\mathbb{R}_{S[L]}$. Before we do this, we need to introduce a norm on the Grassmann algebra:

Definition 3.2. *Suppose* X *is an element of* $\mathbb{R}_{S[L]}$*, with*

$$X = \sum_{\underline{\lambda} \in M_L} X_{\underline{\lambda}} \beta_{[\underline{\lambda}]}.$$

We define $\|X\|$ *by*

$$\|X\| = \sum_{\underline{\lambda} \in M_L} |X_\lambda|.$$

Now, we can state the definition of a G^∞ map on $\mathbb{R}^{m,n}_{S[L]}$.

Definition 3.3. *Let* U *be an open set in* $\mathbb{R}^{m,n}_{S[L]}$ *and* $f : U \to \mathbb{R}_{S[L]}$*. Then*

- f *is said to be* G^0 *if* f *is continuous on* U *with respect to the usual finite-dimensional vector space topology.*
- f *is said to be* G^1 *on* U *if there exist* m *continuous functions* $\partial^E_i f : U \to \mathbb{R}_{S[L]}, i = 1, ..., m,$ *and* n *continuous functions* $\partial^O_j f : U \to \mathbb{R}_{S[L]}, j = 1, ..., n$ *and a function* $\rho : \mathbb{R}^{m,n}_{S[L]} \to \mathbb{R}_{S[L]}$ *which satisfies*

  $$\|\rho(h; \eta)\| \to 0 \text{ as } \|(h; \eta)\| \to 0$$

  such that, if $(x; \eta)$ *and* $(x + h, \xi + \eta)$ *are both in* U*, then*

  $$f(x + h; \xi + \eta) = f(x; \xi) + \sum_{i=1}^{m} h^i (\partial^E_i f)(x; \xi) + \sum_{j=1}^{n} \eta^j (\partial^O_j f)(x; \xi) + \|(x; \xi)\| \rho(h; \eta).$$

- *The definition of* G^p*, where* p *is a finite positive integer, is made inductively. A function* f *is said to be* G^p *on* U *if* f *is* G^1 *on* U *and it is possible to choose* $\partial^S_k f, k = 1, ..., m + n$ *which are* G^{p-1} *on* U*. (Here* ∂^S_k *denotes a partial derivative of either parity, with* $\partial^S_i = \partial^E_i$ *for* $i = 1, ..., m$ *and* $\partial^S_{j+m} = \partial^O_j$ *for* $j = 1, ..., n$*).*
- f *is said to be* G^∞ *on* U *if it is* G^p *for any positive integer* p*.*
- f *is said to be* G^ω *on* U *if, given any point* $X = (x; \xi)$ *in* U*, there exists a neighborhood* N_X *of* X *such that, for all* $Y = (y, v)$ *in* N_X*,* $f(Y)$ *is equal to the sum of an absolutely convergent power series of this form:*

  $$f(X) = \sum_{k_1=0,...,k_{m+n}=0}^{\infty} a_{k_1...k_{m+n}} (Y^1 - X^1)^{k_1} ... (Y^{m+n} - X^{m+n})^{k_{m+n}}$$

  (with each coefficient $a_{k_1...k_{m+n}}$ *in* $\mathbb{R}_{S[L]}$*).*

- *The set of G^p functions of U into $\mathbb{R}_{S[L]}$ is denoted $G^p(U)$.*

Also, suppose that $g : U \to \mathbb{R}^{r,s}_{S[L]}$. Then g is said to be G^∞ on U if each of the $r+s$ components of g is G^∞. The set of all functions is denoted $G^\infty(U, \mathbb{R}^{r,s}_{S[L]})$.

The definition of a G^∞-function is stricter than that of a C^∞-function: all G^∞ functions are C^∞, but not all C^∞ functions are G^∞. Thus, it makes sense to classify G^∞ functions in terms of C^∞ functions. This is the idea behind Grassmann analytic continuation:

Definition 3.4. *Suppose that V is an open subset of $\mathbb{R}^m$ and U is a subset of $\mathbb{R}^{m,n}_{S[L]}$ such that $\epsilon_{m,n}(U) = V$. Then $\widehat{f}$ is called the **Grassmann analytic continuation** of f, given by*

$$\widehat{\ }: C^\infty(V, \mathbb{R}_{S[L]}) \to \{\text{functions of } U \text{ into } \mathbb{R}_{S[L]}\}$$

via

$$\widehat{f}(x;\epsilon) = \sum_{i_1=0,\dots,i_m=0}^{L} \frac{1}{i_1!\dots i_m!} \partial_1^{i_1} \dots \partial_m^{i_m} f(\epsilon_{m,n}(x)) \times s(x^1)^{i_1} \dots s(x^m)^{i_m}$$

Now that we have the definition, let's see why it's useful:

Theorem 3.5. *Given a function f in $G^\infty(U)$, there exist functions $f_{\underline{\mu}}$ in $C^\infty(\epsilon_{m,n}(U)), \underline{\mu} \in M_n$ such that*

$$f = \sum_{\underline{\mu} \in M_n} \widehat{f_{\underline{\mu}}} \xi^{\underline{\mu}}$$

where ξ^j (for $j = 1, \dots, n$) are the odd coordinate functions $\xi^j(x;\xi) = \xi^j$ and $\xi^{\underline{\mu}} = \xi^{\mu_1} \dots \xi^{\mu_k}$ if $\underline{\mu} = \mu_1 \dots \mu_k$.

Roughly, this gives us a way to study G^∞ functions in terms of C^∞ functions.

3.2 G^∞ Functions on $\mathbb{R}^{m,n}_S$

However, there is a major flaw with G^∞ functions on $\mathbb{R}^{m,n}_{S[L]}$: since $\mathbb{R}_{S[L]}$ is not a field, the odd partial derivatives $\partial^O_j f, j = 1, \dots, n$ will not in general be unique. This is precisely why we introduced $\mathbb{R}_S$ and modeled our supermanifolds on $\mathbb{R}^{m,n}_S$ instead of on $\mathbb{R}^{m,n}_{S[L]}$. Using an infinite-dimensional Grassmann algebra, there are no longer any elements that are annihilated by an arbitrary odd element of the algebra, making the odd derivatives well-defined.

Now that we understand why we introduced $\mathbb{R}^{m,n}_S$, let's develop the same theory from the previous section, but on the "better" space.

Definition 3.6. *Let V be open in $\mathbb{R}^m$ and $f : V \to \mathbb{R}_S$.*

- *f is said to be C^∞ if for each positive integer L, the function $\mathcal{P}_L \circ f : V \to \mathbb{R}_{S[L]}$ is C^∞, where $\mathcal{P}_L$ denotes the projection of $\mathbb{R}_S$ onto $\mathbb{R}_{S[L]}$ obtained by setting all generators $\beta_{[r]}$ with $r > L$ to zero. The set of all functions is denoted $C^\infty(V, \mathbb{R}_S)$.*
- *If f is a function in $C^\infty(V, \mathbb{R}_S)$, then the function $\widehat{f} : (\epsilon_{m,0})^{-1}(V) \to \mathbb{R}_S$ is defined by*

$$\widehat{f}(x;\epsilon) = \sum_{i_1=0,\dots,i_m=0}^{L} \frac{1}{i_1!\dots i_m!} \partial_1^{i_1} \dots \partial_m^{i_m} f(\epsilon_{m,n}(x)) \times s(x^1)^{i_1} \dots s(x^m)^{i_m}$$

Similar to previously, let's use the Grassmann analytic expansion, which helps confirm that our function is indeed a superfunction (compare with Definition 3.1).

Definition 3.7. *Let U be open in $\mathbb{R}^{m,n}_S$. Then $f : U \to \mathbb{R}_S$ is G^∞ on U if and only if there exists a collection $\{f_{\underline{\mu}} | \underline{\mu} \in M_n\}$ of functions in $C^\infty(\epsilon_{m,n}(U))$ such that*

$$f(x;\xi) = \sum_{\underline{\mu} \in M_n} \widehat{f_{\underline{\mu}}}(x) \xi^{\underline{\mu}}$$

for each $(x;\xi)$ *in* U*. This expansion is called the* ***Grassmann analytic expansion*** *of* f *and the functions* $f_{\underline{\mu}}$ *are called the Grassmann analytic coefficients of* f*.*

If a function f is G^∞, the notation suggests that f should be G-differentiable infinitely many times. What exactly is the derivative here? Let's write it explicitly:

Definition 3.8. *Suppose that* f *is in* $G^\infty(U)$ *with Grassmann expansion coefficients* $f_{\underline{\mu}}$*. Then for* $i = 1, ..., m$*, the derivative* $\partial_i^E f$ *is defined to be the function in* $G^\infty(U)$ *with Grassmann expansion coefficients* $\partial_i f_{\underline{\mu}}$ *while for* $j = 1, ..., n$*, the derivative* $\partial_j^O f$ *is defined by*

$$\partial_j^O f(x;\xi) = \sum_{\underline{\mu} \in M_n} (-1)^{|f_{\underline{\mu}}|} p_{j,\underline{\mu}} \widehat{f_{\underline{\mu}}}((x)\xi^{\underline{\mu}/j})$$

where

$$p_{j,\underline{\mu}} = \begin{cases} (-1)^{\ell+1} & \text{if } j = \mu_\ell \\ 0 & \text{otherwise} \end{cases}$$

and

$$\underline{\mu}/j = \begin{cases} \mu_1 ... \mu_{\ell-1}\mu_{\ell+1} ... \mu_k & \text{if } j = \mu_\ell \\ 0 & \text{otherwise} \end{cases}$$

3.3 G^∞ DeWitt Supermanifolds via G^∞-Charts

Next, we introduce charts and atlases for G^∞ DeWitt supermanifolds. Similar to the classical case, a G^∞-chart allows us to define coordinates on a supermanifold such that the transition functions between overlapping charts preserve the G^∞ structure.

Definition 3.9. *Let* $\mathcal{M}$ *be a set, and let* m *and* n *be positive integers:*

- *An* $(m,n) - G^\infty$*-chart on* $\mathcal{M}$ *is a pair* (V, ψ) *where* V *is a subset of* $\mathcal{M}$ *and* ψ *is a bijective mapping of* V *onto an open subset of* $\mathbb{R}_S^{m,n}$ *(in the DeWitt topology).*
- *An* $(m,n) - G^\infty$ *atlas on* $\mathcal{M}$ *is a collection of charts* $\{(V_\alpha, \psi_\alpha) | \alpha \in \Lambda\}$ *such that*
  - $\bigcup_{\alpha \in \Lambda} V_\alpha = \mathcal{M}$
  - *for each* α, β *in* Λ *such that* $V_\alpha \cap V_\beta \neq \emptyset$*, the map*

    $$\psi_\alpha \circ \psi_\beta^{-1} : \psi_\beta(V_\alpha \cap V_\beta) \to \psi_\alpha(V_\alpha \cap V_\beta)$$

    is G^∞*.*
- *An* $(m,n) - G^\infty$ *atlas* $\{(V_\alpha, \psi_\alpha) | \alpha \in \Lambda\}$ *on* $\mathcal{M}$ *which is not contained in any other such atlas on* $\mathcal{M}$ *is called a complete* $(m,n) - G^\infty$ *atlas on* $\mathcal{M}$
- *An* $(m,n) - G^\infty$ *DeWitt supermanifold consists of a set* $\mathcal{M}$ *together with a complete* $(m,n) - G^\infty$ *atlas on* $\mathcal{M}$*.*

Finally, we can establish that the collection of sets defined by these charts forms a topology on the supermanifold.

Theorem 3.10. *Let* $\mathcal{M}$ *be a* $(m,n) - G^\infty$ *DeWitt supermanifold with complete atlas* $\{(V_\alpha, \psi_\alpha) | \alpha \in \Lambda\}$*. Let* Γ_{DeWitt} *be the collection of subsets of* $\mathcal{M}$ *consisting of sets* $U \subset \mathcal{M}$ *such that, for all* $\alpha \in \Lambda, \psi_\alpha(U \cap V_\alpha)$ *is open in* $\mathbb{R}_S^{m,n}$ *with the DeWitt topology. Then* Γ_{DeWitt} *is a topology on* $\mathcal{M}$*.*

Earlier, we defined the DeWitt topology on the superspace $\mathbb{R}_S^{m,n}$. Now, we have "extended" this topology onto the supermanifold $\mathcal{M}$. This topology provides the structure necessary for defining and studying smooth functions, differential forms, and other geometric structures on superspaces, thereby extending the framework of classical differential geometry to the supergeometric setting.

3.4 Manifolds Underlying Supermanifolds

By now, it should be clear that not all supermanifolds are manifolds. But there is a neat way to recover a manifold from a supermanifold:

Definition 3.11. *Let $\mathcal{M}$ be a G^∞ DeWitt supermanifold with atlas $\{(V_\alpha, \psi_\alpha)|\alpha \in \Lambda\}$. Then*

- *The relation $\sim$ defined on $\mathcal{M}$ by $p \sim q$ if and only if there exists $\alpha \in \Lambda$ such that both p and q lie in V_α and*

$$\epsilon_{m,n}(\psi_\alpha(p)) = \epsilon_{m,n}(\psi_\alpha(q))$$

  is an equivalence relation.

- *The space $\mathcal{M}_{[\emptyset]} = \mathcal{M}/\sim$ has the structure of an m-dimensional C^∞ manifold with atlas $\{(V_{[\emptyset]\alpha}, \psi_{[\emptyset]\alpha})|\alpha \in \Lambda\}$, where*

$$\begin{aligned} V_{[\emptyset]\alpha} &= \{[p]|p \in V_\alpha\} \\ \psi_{[\emptyset]\alpha} &: V_{[\emptyset]\alpha} \to \mathbb{R}^m \\ [p] &\mapsto \epsilon_{m,n} \circ \psi_\alpha(p) \end{aligned}$$

  where $[]$ denotes equivalence classes in $\mathcal{M}$ under $\sim$.

The manifold $\mathcal{M}/\sim$ is called the ***body*** *of $\mathcal{M}$ and is denoted $M_{[\emptyset]}$.*

This shows that we can go "full circle":

$$\text{manifold } M \to \text{supermanifold } \mathcal{M} \to \text{manifold } M_{[\emptyset]}$$

but in general, the body $M_{[\emptyset]}$ is not the same manifold as M.

3.5 Example: The Real Super Projective Space $\mathbb{SRP}^{m,n}$

To illustrate the concept of G^∞ DeWitt supermanifolds, we now consider the real super projective space, which serves as a concrete example. This space is constructed by taking the quotient of a certain open subset of a superspace and imposing an equivalence relation on it. Let us define this space more formally.

Definition 3.12. *We define the following:*

- *Let $U \subset \mathbb{R}_S^{m+1,n}$ be the set $(\epsilon_{m+1,n})^{-1}(\mathbb{R}^{m+1} - \{0\})$ where*

$$\begin{aligned} \epsilon_{m,n} &: \mathbb{R}^{m,n}_{S[L]} \to \mathbb{R}^m \\ (x^1, ..., x^m; \xi^1, ..., \xi^n) &\mapsto (\epsilon(x^1), ..., \epsilon(x^n)) \end{aligned}$$

- *Let $\sim$ be the equivalence relation on U with $(x;\xi) \sim (y;\eta)$ if and only if there exists an invertible even element ℓ of $\mathbb{R}_S$ such that*

$$\begin{aligned} x^i &= \ell y^i \qquad i = 1, ..., m \\ \xi^j &= \ell \eta^j \qquad j = 1, ..., n \end{aligned}$$

We call $\mathbb{SRP}^{m,n} = U/\sim$ the ***real super projective space****.*

The construction of the real super projective space is now complete, and we proceed to verify that this space is indeed a $(m,n) - G^\infty$ DeWitt supermanifold.

Proposition 3.13. *$\mathbb{SRP}^{m,n}$ is a $(m,n) - G^\infty$ DeWitt supermanifold.*

Proof. An atlas is given as follows: for $i = 1, ..., m = +1$, let

$$V_i = \{[(x;\xi)]|(x;\xi) \in U, \epsilon(x^i) \neq 0\}$$

where $[(x;\xi)]$ is the equivalence class containing the point $(x;\xi)$.

The transition functions ψ are given by

$$\begin{aligned}\psi_i : V_i &\to \mathbb{R}_S^{m,n}\\ [(x;\xi)] &\mapsto \left(\frac{x^1}{x^i},...,\frac{\widehat{x^i}}{x^i},...,\frac{x^{m+1}}{x^i},\frac{\xi^1}{x^i},...,\frac{\xi^n}{x^i}\right)\end{aligned}$$

where $\widehat{x^i}$ denotes ommision. The transition functions are well-defined because if $(x;\xi) \sim (y;\eta)$ and $\epsilon(x^i) \neq 0$, then $\frac{x^j}{x^i} = \frac{y^j}{y^i}$ for $j = 1,...,m+1$ and $\frac{\xi^j}{x^i} = \frac{\eta^j}{y^i}$ for $j = 1,...,n$.

It is straightforward to check that the transition functions are G^∞. We will check it only for $(m,n) = (2,1)$ since the general case is basically the same computation but messier. We have

$$\phi_1([(x^1,x^2;\xi)]) = \left(\frac{x^2}{x^1},\frac{\xi}{x^1}\right), \quad \phi_2([(x^1,x^2;\xi)]) = \left(\frac{x^1}{x^2},\frac{\xi}{x^2}\right)$$

and

$$\phi_1^{-1}\left(\frac{x^2}{x^1},\frac{\xi}{x^1}\right) = (x^1,x^2,\xi), \quad \phi_2^{-1}\left(\frac{x^1}{x^2},\frac{\xi}{x^2}\right) = (x^1,x^2,\xi)$$

so the ϕ_{ij} are given by

$$\phi_{12}\left(\frac{x^2}{x^1},\frac{\xi}{x^1}\right) = \left(\frac{x^1}{x^2},\frac{\xi}{x^2}\right), \quad \phi_{21}\left(\frac{x^1}{x^2},\frac{\xi}{x^2}\right) = \left(\frac{x^2}{x^1},\frac{\xi}{x^1}\right).$$

We can write all four of these using a Grassmann analytic expansion with coefficients:

$$\widehat{f}_1\left(\frac{x^2}{x^1}\right) = \frac{x^1}{x^2}, \qquad \widehat{f}_1\left(\frac{x^1}{x^2}\right) = \frac{x^2}{x^1}, \qquad \widehat{f}_2\left(\frac{x^2}{x^1}\right) = \frac{\xi}{x^2}, \qquad \widehat{f}_2\left(\frac{x^1}{x^2}\right) = \frac{\xi}{x^1}$$

which are all clearly C^∞ on $\mathbb{R}_S$, so the transition functions are G^∞ and $\mathbb{SRP}^{2,1}$ is a $(2,1) - G^\infty$ DeWitt supermanifold. □

4 What's Next?

We conclude with a brief overview of some current areas of focus for modern supergeometers:

- Lie Supergroups/Superalgebras: The theory of Lie groups extends naturally to Lie supergroups. The research behind Lie supergroups has been largely driven by the mutual interest among mathematicians and physicists about the super Poincaré group: for many mathematicians, supersymmetry is defined as the action of the super Poincaré group on a theory, and for physicists, the super Poincaré group gives the symmetries of supersymmetric quantum field theory. Since the super Poincaré group is itself a Lie supergroup, it has garnered significant attention, along with other Lie supergroups. As with classical Lie groups, classifying these groups is a major challenge, typically addressed through the study of their representations. The classification of simple Lie supergroups was completed by Victor Kac, and ongoing research continues to explore the representations of Lie superalgebras, which remains an active area of research.

- Integration on Supermanifolds: The path integral is one of the most useful concepts in physics, so it follows naturally that we would extend this idea to supermanifolds. However, integration on supermanifolds was the first area where classical and the corresponding "super" theory were developed vastly differently. Today, the Berezin integral is widely regarded as the standard for integration on supermanifolds. But it is quite bizzare: it is algebraic, does not appear to have any elementary properties of a classical integral, resembles a derivative more than an integral, and has no measure-theoretic interpretation. Nonetheless, it manages to be largely both useful and elegant, and many open problems are based on the Berezin integral. For example, one of the most active areas of research is the RMS formalization of superstring perturbation theory, where the theory is formulated in terms of Berezin integration on the moduli space of super Riemann surfaces.

- Noncommutative Supergeometry: Earlier, we discussed commutative superalgebras, which were developed many decades before the more challenging, noncommutative versions began to take shape. With the foundations of Hilbert superspaces, C^*-superalgebras, and quantum supergroups established, the theory has seen significant advancement. These structures now appear in a variety of important contexts, including the deformation quantization of symplectic supermanifolds and the harmonic analysis of Lie supergroups. As a result, they have become central to many areas of research.
- Supergeometric Homotopy Theory: Some homotopy theorists argue that the geometry underlying physics should be supergeometric homotopy theory. Inspired by Grothendieck's works decades earlier, these theorists attempt to apply functional geometry in a higher topos that captures the structure of higher-dimensional superspaces. This emerging theory has many desirable features, including numerous connections with higher Lie theory and M-theory. However, given that this field is still in its early stages, there is not yet a broad consensus on whether it constitutes the most promising direction for future research in supergeometry.

References

[1] Axel de Goursac. Noncommutative supergeometry and quantum supergroups. *Journal of Physics: Conference Series*, 597:012028, April 2015.

[2] Pierre Deligne. *Quantum Fields and strings: A course for mathematicians. vol. 1*. American Mathematical Society, 1999.

[3] Bryce DeWitt. *Supermanifolds*. Cambridge University Press, 1992.

[4] Domenico Fiorenza, Hisham Sati, and Urs Schreiber. The rational higher structure of m-theory: Lms/epsrc durham symposium on higher structures in m-theory. *Fortschritte der Physik*, 67(8-9), May 2019.

[5] Alice Rogers. *Supermanifolds: Theory and applications*. World Scientific, 2007.

[6] V. S. Varadarajan. *Supersymmetry for mathematicians: An introduction*. American Mathematical Society, 2004.

[7] Julius Wess and Jonathan Bagger. *Supersymmetry and supergravity*. Princeton University Press, 1992.

[8] Edward Witten. Notes on supermanifolds and integration, 2016.